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Universality of anisotropic turbulence

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Abstract

We review some ideas about the physics of small-scale turbulent statistics, focusing on the scaling behavior of anisotropic fluctuations. We present results from direct numerical simulations of three-dimensional *homogeneous*, anisotropically forced, turbulent systems: the Rayleigh–Bénard system, the random-Kolmogorov-flow, and a third flow with constant anisotropic energy spectrum at low wave numbers. A comparison of the anisotropic scaling properties displays good similarity among these very different flows. Our findings support the conclusion that scaling exponents of anisotropic fluctuations are *universal*, i.e., independent of the forcing mechanism sustaining turbulence.

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In recent years theoretical, numerical and experimental work has been done to study the statistics of small scales in turbulent flows [1]. The understanding of *universality* has been the central matter of investigation in the scientific community. By universality, we mean to which extent small-scale turbulent fluctuations are statistically independent of the large-scale set-up used to inject energy in the flow. Actually, robustness of small-scale physics cannot be really exact; for instance, different forcings

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generally inject different large-scale anisotropic fluctuations, which might have some direct or indirect influence also on the small-scale statistics. A first strong requirement for universality to hold is that large-scale anisotropic fluctuations become more and more negligible going to smaller and smaller scales. This behavior is indeed always observed in both experiments and numerical simulations [2–7]. Another important question about the universality of small-scales statistic concerns the anisotropic components on their own, independently from their comparison with the isotropic ones. In particular, it is important to understand whether the anisotropic components of any turbulent correlation function possess or not *universal* scaling exponents.

In this paper, we review various attempts to study the small-scale anisotropic behavior by means of numerical simulations. The main focus of our analysis is a comparison between the statistics of the homogeneous Rayleigh–Bénard system (HRB) [8], i.e., a convective cell with fixed linear mean temperature profile along the vertical direction, with that of the random-Kolmogorov-flow (RKF), a flow sustained by a random anisotropic forcing active only at large scales [5,6]. From the comparison, we show that all the investigated systems display almost indistinguishable, in the limit of the numerical resolution, small-scale anisotropic (and isotropic) scalings. In other words we find a high degree of small-scale universality for all measurable anisotropic components. This result is particularly relevant because its validity is possible only if both systems have *anomalous* anisotropic small-scale fluctuations, that is if they do not follow the dimensional predictions that can be derived by the equations of motion.

Before discussing these results we recall the technique of $SO(3)$ decomposition and two dimensional predictions available for the scaling of the HRB and RKF structure functions in the non isotropic sectors. By reviewing the numerical results of the HRB and RKF anisotropic properties, we then show in detail to which extent our findings support the idea of small-scale *universality*.

An homogeneous Rayleigh–Bénard system is a convective cell with fixed linear mean temperature profile along the vertical direction. The flow is obtained by integrating the Boussinesq system in a fully periodic cubic domain, that is imposing three-periodic boundary conditions to the velocity field $\boldsymbol{v}$, and to the fluctuating part of the temperature field T' [9]. For large Rayleigh numbers, $Ra = \alpha\, g\, \Delta T H^3/(v\kappa)$, HRB shows a turbulent convective dynamics with absence of both viscous and thermal boundary layers [9]. Here α is the thermal expansion constant, g is the buoyancy coefficient, $\Delta T/H$ is the mean temperature gradient imposed onto the system, H is the height of the convective cell, while v and κ are the viscosity and the diffusivity coefficients, respectively. The typical length characterizing the forcing mechanism in a convective system is the so-called Bolgiano scale [10], defined as $L_B \equiv \epsilon^{5/4} N^{-3/4} (\alpha g)^{-3/2}$, where ϵ and N are the energy and temperature dissipations. A classical reasoning relies on the idea that the turbulent dynamics below the Bolgiano length is unaffected by buoyancy effects, while above the Bolgiano scale it might be strongly influenced by the temperature fluctuations, because of their active feedback on the velocity field. In the HRB system, L_B is of the order of the integral scale H of the cell, hence temperature fluctuations may have a leading role only at the largest scales in the system [8,11].

The main advantage of the HRB system is that the intrinsic homogeneity along the three directions allows for a systematic study of the scaling properties without spurious (non-homogeneous) effects, always present in standard Rayleigh–Bénard systems with boundary layers.

Direct numerical simulations of the HRB cell are performed using a lattice Boltzmann scheme, with spatial resolution of 240^3. The Prandtl number is equal to unit, the Rayleigh number is $Ra = 1.38 \times 10^7$, and the Taylor-scale Reynolds number is $Re_\lambda \sim 100$.

The random-Kolmogorov is also a fully periodic flow. The large-scale anisotropic random forcing, pointing in the direction $\hat{z}$, has a spatial dependency only on the x coordinate and it is different from zero at the two wave numbers: $\boldsymbol{k}_1 = (1,0,0), \boldsymbol{k}_2 = (2,0,0)$. Namely, $\boldsymbol{f}_i(\boldsymbol{k}_{\{1,2\}}) = \delta_{i,3} f_{\{1,2\}} \exp(\theta_{\{1,2\}})$, where $f_{\{1,2\}}$ are fixed amplitudes and $\theta_{\{1,2\}}$ are independent random phases, δ-correlated in time. Random phases assure an almost homogeneous statistics, without destroying the high anisotropy introduced by the energy injection mechanism. The RKF was simulated using a pseudo-spectral code at resolution 128^3 and 256^3. Taylor-scale Reynolds numbers are $Re_\lambda = 60$ and $Re_\lambda = 90$, respectively. Energy is dissipated by a hyper-viscous term at small scales.

A third set of simulations (flow 3), is performed by keeping the total energy of the flow constant on a subset of Fourier modes lying in a plane $\boldsymbol{k} = (k_x, k_y, 0)$, with $|\boldsymbol{k}| < 4$. When discussing the universality of anisotropic scaling, we also review results obtained with this different anisotropic system.

As mentioned before, anisotropy can be studied by means of decomposition of physical observables onto the irreducible representations of the rotational symmetry group $SO(3)$ [12]. In particular we are interested in the $SO(3)$ decomposition of scalar quantities, because of their simple representation. This is the case of velocity longitudinal structure functions, $S^{(p)}(\boldsymbol{r}) = \langle [(\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{v}(0)) \cdot \hat{\boldsymbol{r}}]^p \rangle$, whose decomposition is just

$$S^{(p)}(\boldsymbol{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_{jm}^{(p)}(r) Y_{jm}(\hat{\boldsymbol{r}}) , \tag{1}$$

where the indices (j,m) of the spherical harmonics $Y_{jm}(\hat{\boldsymbol{r}})$ label the total angular momentum and its projection on a reference axis, respectively (see Refs. [5,12] for more details). As for the statistical properties in the inertial range of scales, the physics is hidden in the projections $S_{jm}^{(p)}(r)$. We are interested in measuring the scaling exponents (if any) of each projection belonging to a different anisotropic sector: $S_{jm}^{(p)}(r) \sim c_{jm} r^{\xi^j(p)}$. Here we have implicitly assumed, on the basis of theoretical results obtained in Ref. [12], that the scaling exponents do not depend on the m index. We notice that velocity structure functions have even parity with respect to $\boldsymbol{r}$, therefore projections with odd j indices vanish. For the Rayleigh–Bénard problem we consider also mixed velocity and temperature structure functions which have, on the other hand, dominant odd parity.

Back to the universality issue, we expect that the coefficients c_{jm} are strongly dependent on the anisotropic properties of the large-scale physics, while the values of the scaling exponents, $\xi^j(p)$, should be independent of the large-scale forcing and/or boundary conditions, thus enjoying a much higher degree of universality. Such a picture can be proved on a rigorous basis for linear problems of scalar/vector advection by Gaussian, white-in-time, velocity fields (Kraichnan models [13]), but their validity is thought to go beyond this class of stochastic models [14].

To obtain a dimensional prediction for the scaling of the anisotropic projections of velocity structure functions the starting point is the stationary equation for the second order structure function (the Kármán–Howarth equation): we decompose it into the various j sectors, paying attention to the tensorial properties of each term appearing in the equation. This gives, in the case of the HRB system, the following matching relation, valid for any anisotropic sectors ($j > 0$) and any order p (see Ref. [8] for details of the derivation),

$$S_{jm}^{(p)}(r) \sim rB_{j-1,m}^{(p-2,1)}(r) \quad \text{dimensional prediction for HRB} . \tag{2}$$

In the previous expression, $B_{j,m}^{(q,1)}(r)$ is the $SO(3)$ projection of the buoyancy-like terms built with q velocity longitudinal increments and one temperature increment, $B^{(q,1)}(\boldsymbol{r}) = \langle [(\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{v}(0)) \cdot \hat{\boldsymbol{r}}]^q (T(\boldsymbol{r}) - T(0)) \rangle$.

Eq. (2) is the simplest *dimensional prediction*, sector by sector, that we can derive for this system consistently with the anisotropic properties of the buoyancy term. It plays a key role in our analysis because, as we show in the following, the observed anisotropic scaling measured in the DNS of the Rayleigh–Bénard system differ from the dimensional prediction (2), i.e., HRB has anomalous anisotropic scaling exponents. We notice also that in the isotropic sector, labeled as $j = 0$, due to the large size of the Bolgiano length, the expected dimensional scaling is the classical Kolmogorov 1941 (K41) $\xi(p) = p/3$.

A similar procedure can be pursued to derive a dimensional scaling prediction for the RKF,

$$S_{jm}^{(p)}(r) \sim r\mathscr{S}\, S_{j-2,m}^{(p-1)}(r) \quad \text{dimensional prediction for RKF} . \tag{3}$$

Here the contribution of the shear term $\mathscr{S}_{ik} \equiv \partial_i U_k$, with U_k being the intensity of the mean velocity, is measured in the $j = 2$ sector in terms of the shear intensity $\mathscr{S}$ (see Ref. [5] for the detailed derivation). Relation (3), made explicit for the scaling exponents, gives $\xi^j(p) = (p + j)/3$; it is worth noticing that, in the isotropic sectors, it corresponds to the classical Kolmogorov 1941 dimensional prediction.

Coming to our numerical results, we first stress that both systems display the same isotropic scaling exponents for structure functions of any order p between 2 and 6, although with different extension of the inertial range, as expected. These are also in agreement with the anomalous exponents measured for homogeneous isotropic turbulence [15,16].

As for the non-isotropic sectors, it was shown in Ref. [8] that HRB exhibits anomalous scaling, i.e., exponents do deviate from prediction (2).

In Ref. [5] it has been shown that also the RKF has an anomalous behavior in the anisotropic sectors: deviations from the expected dimensional scaling exponents, $\xi^j(p) = (p + j)/3$, have been observed for the structure function projections of any measured order p and sector j, precisely with $p \in [2, 6]$ and $j \in [2, 6]$.

On the basis of these observations, we now discuss the question of the universality of small-scale turbulent fluctuations, comparing the two previous systems with the results of the DNS of a third flow with different large-scale anisotropic forcing (flow 3). First in Fig. 1, we plot the projections of the most important anisotropic contributions for the second and fourth order structure functions of RKF and flow 3,

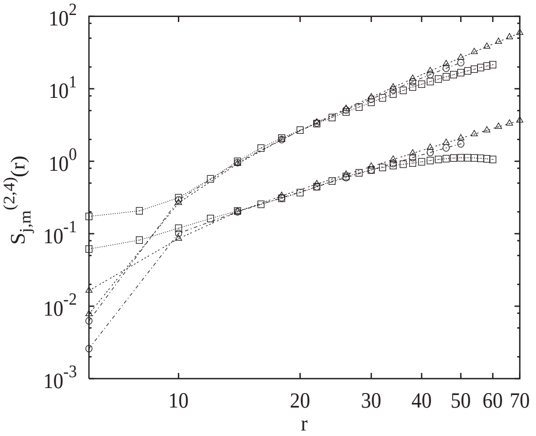

Fig. 1. Log–log plot of the most intense anisotropic sectors for structure functions of order 2 and 4, at different Reynolds numbers and different large-scale forcing. Top three curves: $S^4_{2,2}(r)$ at $Re_\lambda = 60$ (○) and $Re_\lambda = 90$ (△) for the RKF, and for flow 3 (see text) at $Re_\lambda = 90$ (□). Bottom curves, $S^2_{4,0}(r)$ for the same three cases.

at two different Reynolds numbers. This test shows the robustness of small-scale fluctuations at changing the energy injection at the large scales. As it can be seen, despite the fact that the anisotropic contributions of flow 3 are much noisier at large scales, we have a quite good qualitative agreement for the scaling properties in the inertial range. This is a first indication that anisotropic scaling might be universal.

A stronger test on universality can be performed using the data of HRB system. Here, not only anomalous anisotropic scaling has been measured but also the statistical behavior is found to be indistinguishable from that observed in the RKF [5]. This point is far from being trivial and should not be underestimated. The HRB has an anisotropic forcing associated to the buoyancy term, which acts at all scales $\sim g\hat{z}\delta T(r)$: this means that there is a direct anisotropic energy injection down to the small scales, at variance with the RKF where the forcing is limited to the large scales. In Fig. 2, a comparison is made of the projections $S^{(p)}_{jm}(r)$ of the Rayleigh–Bénard cell with those of the RKF, for $j = 4$ and $m = 0, 2$. The sector $j = 2$ is not shown because, as noticed in Ref. [5], the $j = 2$ data in the RKF are spoiled by a sign inversion, which makes the comparison inconclusive. Moreover in RKF, at the resolution of both simulations, the $j = 4$ sector is unexpectedly more intense than the $j = 2$ sector at all scales.

From Fig. 2, the agreement is quite satisfactory, except for scales smaller than the viscous length, where as usual the $SO(3)$ decomposition suffers interpolation errors. The small discrepancies at large scales are also to be expected: the inertial properties of the two flows have to match with quite different conditions at large scales. Concerning the dependency on the order of the moment p, for a fixed anisotropic sector j, we notice that there is a tendency toward saturation, i.e., the higher is the anisotropic j sector, the smaller is the increase of the scaling exponents $\xi^j(p)$ as a function of p. The same comparison for $j = 6$ (not shown) qualitatively fits with the other results.

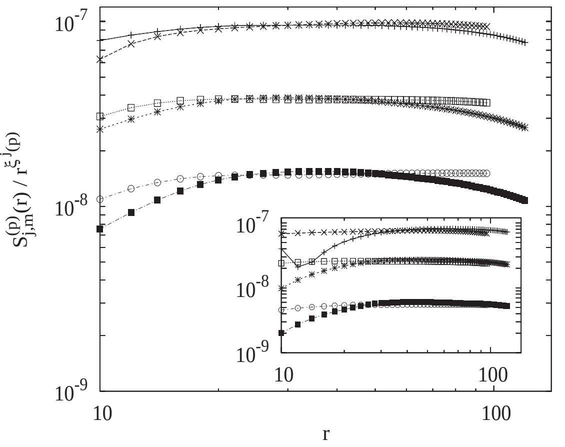

Fig. 2. Log–log plot of compensated anisotropic $j=4, m=0$ projections $S^{(p)}_{4,0}(r)/r^{\xi^4(p)}$ vs. r, for HRB and RKF flows. Top curves refer to $p=2$: the best fit exponents which compensate HRB and RKF curves are $\xi^4(2)=1.7$ and $\xi^4(2)=1.66$, respectively. Curves in the middle refer to the same quantities but for $p=4$: compensation has been obtained with $\xi^4(4)=2.05$ for HRB, and $\xi^4(4)=2.2$ for RKF. Bottom curves refer to $p=6$: here $\xi^4(6)=2.3$ for HRB, and $\xi^4(6)=2.5$ for RKF. Notice that the curves of the two flows are compensated with very similar values of the exponents (within 10%). Inset: the same but for $j=4, m=2$, compensation has been done with the same values used for $j=4, m=0$, to show the Independence of the scaling exponents from m.

Some comments are worth to be done. The independence of the small-scale statistics from the m value implies independence from the external forcing which breaks rotational invariance. Indeed, the unforced equation for the projections $S^{(p)}_{j,m}(r)$ are independent of the index m: so, the fact that the scaling exponents $\xi_j(p)$ are m independent tell us, indirectly, that the forcing does not influence the inertial range statistics.

In addition, the fact that inertial scale fluctuations of the HRB and of the RKF are almost indistinguishable is the first important confirmation of the universality of anisotropic fluctuations, with respect to the external forcing, in sectors with $j=4,6$. Similar conclusions can be drawn for the $j=2$ sector in different experimental set-up [2–4], i.e., in the only sector where scaling has been indirectly measured in experiments.

We conclude by summarizing the main results of our analysis. First, anisotropic fluctuations in large-scale forced systems and Rayleigh–Bénard convective cells are *anomalous*, i.e., they do not follow the predictions based on purely dimensional assumptions. Second, notwithstanding the direct influence of the HRB forcing at intermediate and small scales, anisotropic fluctuations are *universal*, i.e., the small-scale dynamics is dominated by anomalous fluctuations, coming from the self-organization of the inertial evolution. This is the mechanism at the origin of anomalous scaling in Kraichnan models of passive/vector advection [13], whose validity, we argue, is to be extended beyond the framework of the linear stochastic models. In the case of Kraichnan models, one rigorously connects the anomalous inertial scaling with the existence of *zero-modes* of the inertial operator [17–19]. Here, for Navier–Stokes equation, we may only stress the striking similarities, without being able to prove our statements rigorously. Clearly more extensive tests at higher Reynolds numbers and with different anisotropic forcings have to be performed in order to draw a firm and possibly quantitative conclusion about the addressed question.

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