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Prandtl number scaling laws in the Homogeneous Rayleigh-Bénard system

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1 Introduction

The scaling behavior of the Nusselt (Nu) number versus the Rayleigh (Ra) and the Prandtl (Pr) numbers in a Rayleigh-Bénard cell has been extensively investigated in recent years. The $Nu(Ra)$ dependency at large Ra values has been studied by experimental works, seeking the so called asymptotic Kraichnan scaling regime, $Nu \sim Ra^{1/2}$ [1], but the problem is still open.
Even less clear is the $Nu(Pr)$ dependence at very high Ra values. Different predictions exist, ranging from negative scaling exponents when $Pr \gtrsim 1$ to the positive large exponent $Nu \sim Pr^{1/2}$, for smaller Pr values. The problem was analyzed theoretically many years ago by Kraichnan [1] and recently by Grossmann and Lohse (G-L) [2]. From the experimental side at moderately large Prandtl values (> 1) a weakly negative exponent dependence has been measured, while measurements and also numerical results at low Pr detect a weak positive scaling exponent, $Nu \sim Pr^{0.14}$ (see [2] for references to the experimental results).
In this paper, we present results from direct numerical simulations of the bulk of a convective cell in a simplyfied model, the Homogeneous Rayleigh-Bénard (HRB) system [3], focusing on the scaling of the Nusselt number, the Reynolds number and the temperature mean value versus Pr.

2 The Homogeneous Rayleigh-Bénard system

The HRB is a cubic fully periodic convective three dimensional cell (of size L), with a fixed linear mean temperature profile along the vertical direction [3],[4]. The temperature fluctuations θ obey the advection equation

$$\partial_t \theta + u_i \partial_i \theta = \chi \partial_i \partial_i \theta + u_3 \Delta T / L. \tag{1}$$

The velocity field $u_i(\boldsymbol{x}, t)$ obeys the standard Boussinesq equation,

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_j^2 u_i + \alpha g \delta_{i3} \theta. \tag{2}$$

where α is the thermal expansion coefficient, g gravitational acceleration, and p the pressure. This model system simulates the bulk of a real Rayleigh-Bénard cell so we expect that it should correctly predict $Nu(Ra, Pr)$ and also $Re(Ra, Pr)$ in the asymptotic regime, where the boundary layer thicknesses are no longer relevant scales for the dynamics. Following the G-L prediction, mean energy and temperature dissipation rates in the bulk of a Rayleigh-Bénard cell can be estimated in terms of the mean wind, the top-bottom thermal gap and the size of the cell. Namely it is expected that in a fully developed turbulent regime the dissipation rates are independent from the transport coefficients ν and χ. These assumptions, combined with the exact relations linking the dissipation rates to global quantities (see [2] for the derivation), give the following scaling relations for the Nusselt and Reynolds numbers

$$Nu \sim Ra^{1/2}\ Pr^{1/2}, \qquad Re \sim Ra^{1/2}\ Pr^{-1/2}. \tag{3}$$

In [3] it has already been shown that $Nu(Ra)$ and $Re(Ra)$ are compatible with the asymptotic $Ra^{1/2}$ scaling, which has been first predicted by Kraichnan in 1962, and also in agreement with G-L. The dependency on Pr in HRB has not been explored sofar.

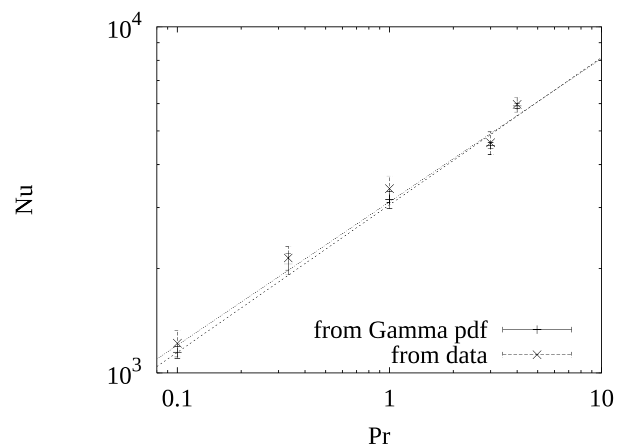


Figure 1: Log-log plot of Nu versus Pr. Two series of point are shown, the first $(-)$ derived from a fit of the *pdf*, the second $(\times)$ directly computed on data. Fitted power laws to the two data sets are also drawn. The fitted exponents in the two case (0.41 ± 0.02 and 0.43 ± 0.02 respectively) coincide within errors.

3 Results and discussion

We run simulations with five different Pr numbers, 1/10, 1/3, 1, 3 and 4, at fixed $Ra = 1.38 \cdot 10^7$ in statistically stationary conditions; the data set collected

ranges between 65 and 155 eddy turn over times. Two procedures to estimate central values and errors has been considered, in the first case central values are obtained from averages over space and time while error bars are estimated using the standard rule $(2\sigma/N)^{1/2}$ where σ is the variance and N the number of eddy turn over times. The second method is based on the fit of the probability density function of the time fluctuations of the physical quantities of interest, now computed on average over space. We adopt the Gamma distribution for this fit, as suggested and discussed in [5]. The two procedures give very similar results.
In Fig.1 we show the behavior of Nu as a function of Pr number, with data evaluated in the two ways described before. Results of power laws fit are also reported. In Fig.2 the same for the Reynolds number is shown. In this case the two fitting procedures yield almost coincident results, and the error bars are remarkably small (for better readability Fig.2 shows only one data set). Our measures support asymptotic predictions in favor of a positive large ($\lesssim 1/2$) Pr-exponent for Nu, and a Reynolds number dependency on Prandtl greater, but very close to the $Re \sim Pr^{-1/2}$ behavior. Additional informations can be

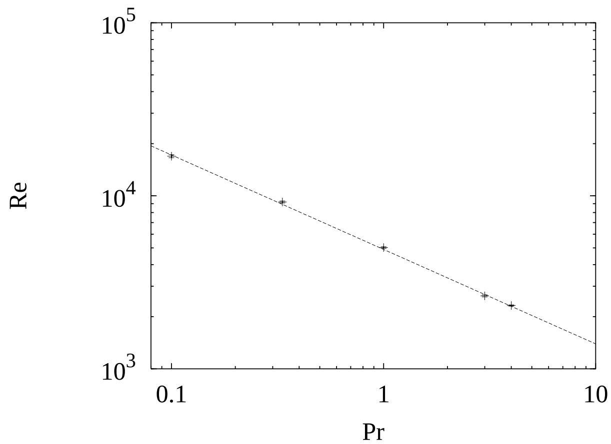


Figure 2: Log-log plot of Re versus Pr. The power-law fit gives an exponent -0.54 ± 0.01.

obtained looking at the θ_{rms} and at temperature and velocity dissipation rates, $\epsilon_\theta = (\chi/2)\langle \sum_i (\partial_i T)^2 \rangle$ and $\epsilon_u = (\nu/2)\langle \sum_{i,j}(\partial_i u_j + \partial_j u_i)^2 \rangle$, whose behavior respect to Pr, as we observed previously, is a key-point in the G-L argument. Fig. 3 shows that the mean thermal fluctuations in the HRB are independent from Pr and equal in size to the background thermal gap (ΔT), the only relevant scale for the temperature field in the system, in agreement with the G-L picture. Concerning dissipation rates, we detect a clear independence from Pr on ϵ_u, while very weak deviations from a constant behavior are present in ϵ_θ.
The question whether the mild deviations from relations (3) observed on HRB

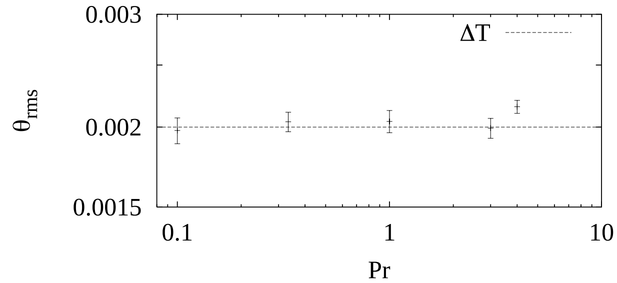
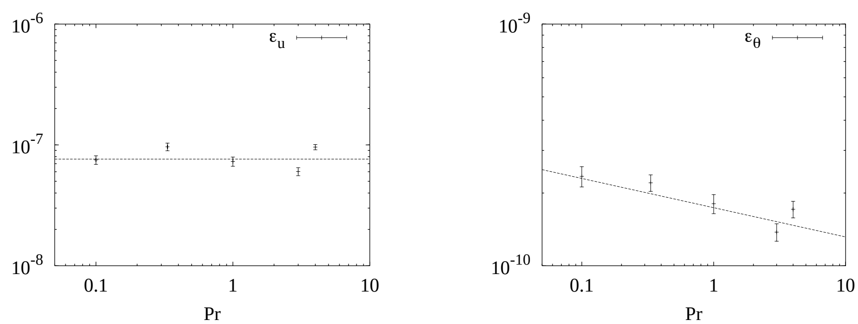


Figure 3: Log-log plot of θ_{rms} versus Pr. The value of the background temperature difference ΔT is shown for comparison.

are connected to non-asymptotic effect, due to the not extremely large value of Ra that we are able to reach in our DNS, or to other more subtle effects deserves, in our opinion, further studies.



(a) (b)

Figure 4: Log-log plot of ϵ_u and ϵ_θ versus Pr. ϵ_u is independent from Pr, while ϵ_θ shows a weakly negative dependency on Pr, -0.12 ± 0.04.

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