Homogeneous Rayleigh-Bénard convection

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1 The problem

Much effort has been expended in recent decades in addressing the problem of heat transfer in Rayleigh-Bénard (RB) thermal convection cells. There is increasing agreement that in general there are no clean scaling laws for $Nu(Ra, Pr)$ and $Re(Ra, Pr)$, apart from asymptotic cases. One of these asymptotic cases has been doped the *ultimate state of thermal convection*, i.e. $Ra \to \infty$, where the heat flux becomes independent of the kinematic viscosity ν and the thermal diffusivity κ. The physics of this regime is that the thermal and kinetic boundary layers have broken down or do not play a role any more for the heat flux and the flow is bulk dominated. Scaling laws for this regime were first suggested by Kraichnan [1], and later by Spiegel [2]. The recent Grossmann-Lohse (GL) theory [3] also gives such an asymptotic regime which is bulk dominated and where the plumes do not play a role , namely

$$Nu \sim Ra^{1/2} Pr^{1/2}, \tag{1}$$

$$Re \sim Ra^{1/2} Pr^{-1/2}. \tag{2}$$

While current experimental data for high Rayleigh numbers are controversial, see [3] for an overview, numerical simulations have not been very effective in studying this regime because of difficulties in dealing with the huge number of degrees of freedom and scale separation engendered when Rayleigh numbers reach the order of at least 10^{12}.

In order to bridge such difficulties, we study a tri-periodic convective cell, or homogeneous Rayleigh-Bénard (HRB) system, to investigate the properties of the convective cell once the effect of boundary layers has been eliminated. A model system such as this was first introduced by Borue and Orszag [4], and studied by means of a spectral DNS (with built-in hyper-viscosity). While these authors focused especially on turbulent spectra and correlation functions behavior, in the present context we address mainly the scaling of integral quantities, such as heat flux and mean velocity fluctuations, respect

to the parameters Ra and Pr.
The physical relevance of the HRB model, in particular the expected similarities and differences with respect to real experiments of fully developed turbulent convection, will be shortly discussed at the end.

2 The model

The system to be studied is described in terms of the following partial differential equations

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u} + \hat{z}\, \alpha g \theta \,, \tag{3}$$

$$\theta_t + \boldsymbol{u} \cdot \nabla \theta = \kappa \Delta \theta + \frac{\Delta T}{L} u_z \,, \tag{4}$$

where $\boldsymbol{u} = (u_x,\, u_y,\, u_z)$ is an incompressible velocity field, $\nabla \cdot \boldsymbol{u} = 0$, ν and κ and αg are respectively the kinematic viscosity, thermal diffusivity and the thermal expansion coefficient times the acceleration due to gravity. These equations are used to describe the evolution of the velocity field in a triply-periodic cubic volume $[0, L]^3$ in the presence of a temperature field $T(\boldsymbol{x}, t) = \overline{T}(\boldsymbol{x}) + \theta(\boldsymbol{x}, t)$. The temperature is expressed as a fluctuation θ with respect to a mean profile $\overline{T}(\boldsymbol{x})$ that is *imposed* to be equal to the mean conductive temperature profile in such a Rayleigh-Bénard cell; i.e. linear and of the form $\overline{T}(\boldsymbol{x}) = -z\Delta T/L + \Delta T/2$.
We performed a high-resolution DNS of the above set of equations, see [6] for details on the implementation, at changing both the Ra and Pr number. Statistical analysis were then performed on the database of flow configurations that were collected, in statistically stationary conditions, over a time interval of order 10^2 large eddy turnover times.

3 Results

In the HRB system the Nusselt number is defined as the dimensionless heat flux

$$Nu = \frac{\langle u_z \theta \rangle}{\kappa \Delta T \; L^{-1}} + 1 \tag{5}$$

where the average $\langle ... \rangle$ is over volume and over time, in statistically stationary conditions. From eqs. (3)-(4) one can derive two exact relations for the volume averaged thermal dissipation rate $\epsilon_\theta = \kappa \left\langle (\partial_i \theta)^2 \right\rangle$ and the volume averaged kinetic dissipation rate $\epsilon_u = \nu \left\langle (\partial_i u_j)^2 \right\rangle$, namely

$$\epsilon_u = \frac{\nu^3}{L^4} Nu Ra Pr^{-2} \tag{6}$$

$$\epsilon_\theta = \kappa \frac{\Delta T^2}{L^2} Nu. \tag{7}$$

One can therefore numerically compute Nu in three different ways: (i) from its direct definition (5), (ii) from the volume averaged kinetic dissipation rate

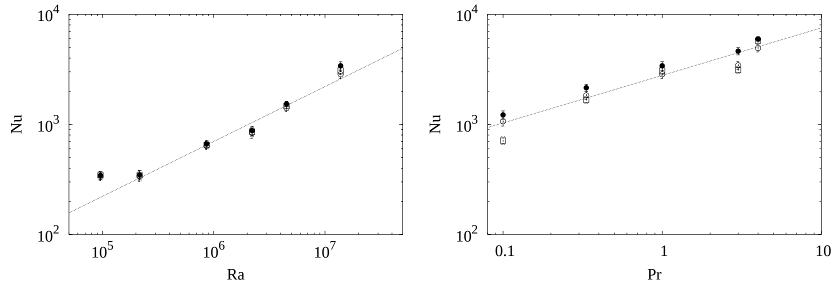

Fig. 1. (a) $Nu(Ra)$ for $Pr = 1$, computed in three different ways: (•) using Eq.(5), (□) using Eq.(6), and (◦) from Eq.(7). The power law fits, performed on the mean value of the three different estimates and for $Ra > 10^5$, gives a slope 0.50 ± 0.05. (b) $Nu(Pr)$ for $Ra = 1.4 \cdot 10^7$, fit performed as before, with a resulting slope of 0.43 ± 0.07.

(6), (iii) from the volume averaged thermal dissipation rate (7).

The results are shown in Figure 1(a) as a function of Ra for $Pr = 1$. There is very good agreement of Nu obtained from the three different methods for all Ra. Fitting all data points beyond $Ra = 10^5$ with an effective power law, we obtain $Nu \sim Ra^{0.50 \pm 0.05}$, consistent with the asymptotically expected law (1). In Figure 1(b) we display Nu as function of Pr for fixed $Ra = 1.4 \cdot 10^7$. For the cases with $Pr \neq 1$ the convergence of the three different methods to calculate Nu is not perfect. This may be due to numerical errors in the resolution of the small scale differences, especially when ν and κ are considerably different. However, one can clearly notice a strong increase of Nu with Pr. A fit with an effective power law gives $Nu \sim Pr^{0.43 \pm 0.07}$, which is again consistent with the asymptotic power law $Nu \sim Pr^{1/2}$ suggested by the GL theory.

Similar conclusions follow from Figure 2, in which we show the Reynolds number, $Re = \frac{u'L}{\nu}$, scaling versus Ra (a) and as function of Pr for fixed Ra (b). We want to stress here that this is the *fluctuation* Reynolds number, defined by the rms velocity fluctuation $u' = \langle \boldsymbol{u}^2 \rangle^{1/2}$: in homogeneous RB no large scale wind exists. $Re(Pr)$ displays an effective scaling law $Re \sim Pr^{-0.55 \pm 0.01}$, consistent with the prediction for the ultimate regime (if one identifies the wind Reynolds number in GL with the fluctuation Reynolds number here).

In conclusion, we confirme that both the Ra- and the Pr-scaling of Nu and Re in homogeneous Rayleigh-Benard convection are consistent with the suggested scaling laws for the bulk-dominated regime.

Although apparently physically unrealizable, because of the boundary conditions, the measures from the highly turbulent HRB model may be tested against a recent series of experimental results obtained from an RB setup especially designed to reduce the influence of top and bottom plates on the physical core of thermal convection [7]. In these experiments the temperature gradient in the bulk of the cell is not imposed but rather, as in fixed flux convection, measured as a dependent parameter. Interestingly these experiments

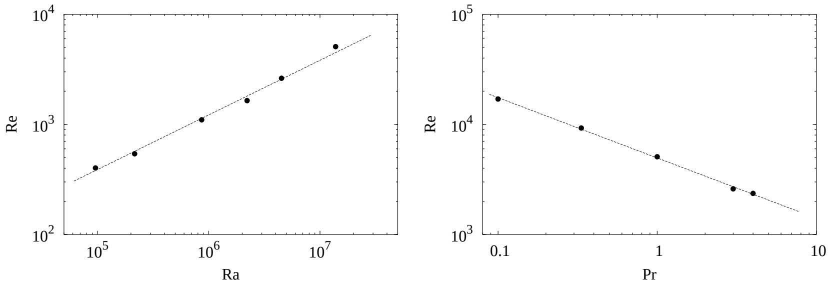

Fig. 2. (a) $Re(Ra)$ for $Pr = 1$, with a fitted slope 0.50 ± 0.02. (b) $Re(Pr)$ for $Ra = 1.4 \cdot 10^7$, with a fitted slope -0.55 ± 0.01.

show, that the Nu and Re versus Ra scalings observed are consistent with our bulk simulations.

On the other hand, a peculiar properties of the HRB model, that becomes particularly relevant to its dynamics in the *low Rayleigh number regime*, is the formation of accelerating antiparallel vertical jets in the system. These flow patterns, that where already observed in the former [4], can be associated to the existence of a particular class of unstable solution for the full non-linear sets of differential equations that we want to introduce in the following [8].

Due to the periodic boundary conditions, the coupled system of equations (3)-(4) admits the particular solution $\theta = \theta_0 e^{\kappa\lambda t}\sin(\boldsymbol{k}\cdot\boldsymbol{x})$, $u_z = u_0 e^{\kappa\lambda t}\sin(\boldsymbol{k}\cdot\boldsymbol{x})$ and $u_x = u_y = 0$, which is independent from the vertical coordinate z (here $\boldsymbol{k} = (k_x, k_y)$) and with:

$$\lambda = -\tfrac{1}{2}\,(Pr+1)k^2 + \tfrac{1}{2}\,\sqrt{(Pr+1)^2k^4 + 4Pr\,(Ra\;L^{-4} - k^4)} \qquad (8)$$

From equation (8) one finds that the first unstable mode appears for $Ra \geq Ra_c = (2\pi)^4 \sim 1558.54$, corresponding to the instability of the smallest possible wavenumber in the system, i.e., $\boldsymbol{k} = \frac{2\pi}{L}(1,0)$.

These solutions are clearly manifest in direct numerical simulations at Rayleigh numbers slightly above the critical value ($Ra \gtrsim Ra_c$) [8] and in general for $Ra \lesssim 10^5$ were only few unstable modes are active [6]. Despite the presence of these exact exploding modes, the system clearly shows that these solutions do not survive indefinitely due to some yet to be explored secondary instability mechanism, resulting always in a statistically stationary behavior. Is the interplay between the active exploding modes and the destabilization that sets the value of the Nusselt number, i.e., the heat transfer through the cell in the low-Ra regime.

References

1. R. H. Kraichnan, (1962) *Phys. Fluids* **5**, 1374.
2. E. A. Spiegel, (1971) *Ann. Rev. Astron. Astrophys.* **9**, 323–352.
3. S. Grossmann and D. Lohse, (2000) *J. Fluid. Mech.* **407**, 27; (2001) *Phys. Rev. Lett.* **86**, 3316; (2004) *Phys. Fluids* **16**, 4462-4472.

4. V. Borue and S. Orszag, (1997) *J. Sci. Comp.* **12**, 305.
5. D. Lohse and F. Toschi, (2003) *Phys. Rev. Lett.* **90,** 034502.
6. E. Calzavarini, D. Lohse, F. Toschi and R. Tripiccione, (2005) *Phys. Fluid* **17**, 055107.
7. M. Gibert, H. Pabiou, F. Chilla and B. Castaing (2005) *private communication and present proceedings.*
8. E. Calzavarini, C. R. Doering, J. D. Gibbon, D. Lohse, A. Tanabe and F. Toschi, (2005) *unpublished, under consideration for publication on Phys. Rev. Lett.*