

$$x - 0,2 \sin x = 0,5 \quad ; x \in [0, 1]; \quad \varepsilon = 0,001$$

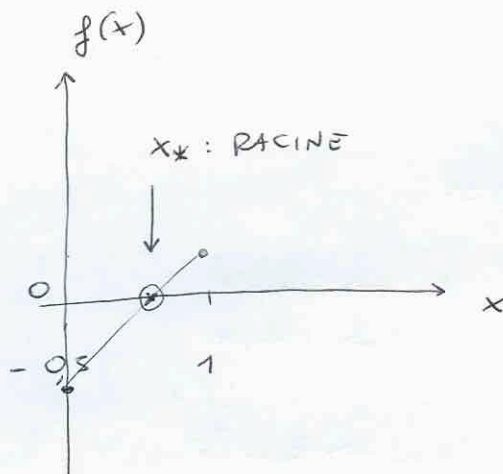
$$f(x) = 0 \quad ; f(x) = x - 0,2 \sin x - 0,5$$

(a)

$$a = 0 \Rightarrow f(a) = a - 0,2 \sin a - 0,5 = -0,5$$

$$b = 1 \Rightarrow f(b) = b - 0,2 \sin b - 0,5 = 1 - 0,5 - 0,2 \sin 1 \approx \underline{\underline{0,84}}$$

$$\approx 0,5 - 0,2 \cdot 0,84 = 0,332$$



$$f(a) \cdot f(b) < 0 \Rightarrow$$

$$\Rightarrow \text{l'INTERVALLE}$$

$$\text{ENCADRE LA SOLUTION}$$

(b) NEWTON - RAPHSON

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = F(x_n)$$

ALGORITHME

$$f'(x) = 1 - 0,2 \cos x$$

$$x_0 = \frac{1}{2} \quad ; \quad \text{CRITÈRE D'ARRÊT : } |x_{n+1} - x_n| < \varepsilon$$

$$x_0 = \frac{1}{2}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \approx 0,616297 ; |x_1 - x_0| \approx 0,116297 > \varepsilon$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0,615468 ; |x_2 - x_1| \approx 0,0008 < \varepsilon$$

$$\rightarrow \boxed{x_* \approx x_2 \approx 0,6154}$$

(c) BISSECTION

ALGORITHME:

$$a_n, b_n, x_n$$

$$x_n = \frac{a_n + b_n}{2}$$

$$\text{si } f(x_n) = 0 \Rightarrow \text{solution}$$

$$\text{si } f(x_n) \cdot f(a_n) > 0 \Rightarrow \begin{aligned} x_n &\rightarrow a_{n+1} \\ b_n &\rightarrow b_{n+1} \end{aligned}$$

sinon

$$\begin{aligned} a_n &\rightarrow a_{n+1} \\ x_n &\rightarrow b_{n+1} \end{aligned}$$

$$a_0 = 0, b_0 = 1, x_0 = \frac{a_0 + b_0}{2} = \frac{1}{2}$$

→ PRELIMINAIREMENT (et pour question (d)) :

$$\varepsilon_1 = 0,001$$

$$I_n = \frac{|a-b|}{2^n}$$

$$I_n < \varepsilon \Rightarrow n = \left\lceil \frac{\log |a-b| - \log \varepsilon}{\log 2} \right\rceil$$

NOMBRE ITÉRATIONS
NÉCESSAIRES

$$\begin{aligned} a=0 \\ b=1 \end{aligned} \rightarrow \left\lceil -\frac{\log \varepsilon}{\log 2} \right\rceil = \left\lceil 9,97 \right\rceil = 9 \div 10 \Rightarrow \boxed{n=10}$$

$$\varepsilon_2 = 0,01 \Rightarrow n = \left\lceil 6,64 \right\rceil = 6 \div 7 \Rightarrow \boxed{n=7}$$

$$\varepsilon_3 = 0,1 \Rightarrow n = \left\lceil 3,32 \right\rceil = 3 \div 4 \Rightarrow \boxed{n=4}$$

$n=0$

$$x_0 = \frac{1}{2} ; a_0 = 0 ; b_0 = 1$$

$$\begin{aligned} f(x_0) &\approx -0,095885 \\ f(a_0) &= -0,5 \end{aligned} \Rightarrow f(x_0) \cdot f(a_0) > 0 \Rightarrow \begin{cases} a_1 \leftarrow x_0 \\ b_1 \leftarrow b_0 \end{cases}$$

 $n=1$

$$a_1 = x_0 = \frac{1}{2} ; b_1 = b_0 = 1 ; x_1 = \frac{a_1 + b_1}{2} = \frac{3}{4} = 0,75$$

$$\begin{aligned} f(x_1) &\approx 0,113672 \\ f(a_1) &\approx -0,095885 \end{aligned} \Rightarrow f(x_1) \cdot f(a_1) < 0 \Rightarrow \begin{cases} a_2 \leftarrow a_1 \\ b_2 \leftarrow x_1 \end{cases}$$

$$|a_1 - b_1| = 0,5 > \varepsilon ;$$

 $n=2$

$$a_2 = a_1 = \frac{1}{2} ; b_2 = x_1 = \frac{3}{4} ; x_2 = \frac{a_2 + b_2}{2} = \frac{5}{8}$$

$$\begin{aligned} f(x_2) &\approx 0,007980 \\ f(a_2) &\approx -0,095885 \end{aligned} \Rightarrow f(x_2) \cdot f(a_2) < 0 \Rightarrow \begin{cases} a_3 \leftarrow a_2 \\ b_3 \leftarrow x_2 \end{cases}$$

$$|a_2 - b_2| = 0,25 > \varepsilon_3$$

 $n=3$

$$a_3 = a_2 = \frac{1}{2} ; b_3 = x_2 = \frac{5}{8} ; x_3 = \frac{a_3 + b_3}{2} = \frac{9}{16}$$

$$\begin{aligned} f(x_3) &\approx -0,04416 \\ f(a_3) &\approx -0,095885 \end{aligned} \Rightarrow f(x_3) \cdot f(a_3) > 0 \Rightarrow \begin{cases} a_4 \leftarrow x_3 \\ b_4 \leftarrow b_3 \end{cases}$$

$$|a_3 - b_3| = \frac{1}{8} > \varepsilon_3$$

 $n=4$

$$a_4 = x_3 = \frac{9}{16} ; b_4 = b_3 = \frac{5}{8} ; x_4 = \frac{a_4 + b_4}{2} = \frac{19}{32} \approx 0,59375$$

$$|a_4 - b_4| = \frac{1}{16} > \varepsilon_2 ; |a_4 - b_4| = 0,0625 < \varepsilon_3$$

$$\begin{aligned} f(x_4) &\approx -0,01845 \\ f(a_4) &\approx -0,04416 \end{aligned} \Rightarrow f(x_4) \cdot f(a_4) > 0 \Rightarrow \begin{cases} a_5 \leftarrow x_4 \\ b_5 \leftarrow b_4 \end{cases}$$

$$\underline{n=5}$$

$$e_5 = x_4 = \frac{19}{32} ; b_5 = b_4 = \frac{5}{8} ; x_5 = \frac{e_5 + b_5}{2} = \frac{39}{64}$$

$$f(x_5) \approx -0,005000$$

$$f(e_5) \approx -0,018145$$

$$\Rightarrow f(x_5) \cdot f(e_5) > 0$$

$$\Rightarrow \begin{cases} e_6 \leftarrow x_5 \\ b_6 \leftarrow b_5 \end{cases}$$

$$|e_5 - b_5| = \frac{1}{32} > \varepsilon_2$$

$$\underline{n=6}$$

$$e_6 = x_5 = \frac{39}{64} ; b_6 = b_5 = \frac{5}{8} ; x_6 = \frac{e_6 + b_6}{2} = \frac{79}{128}$$

$$f(x_6) \approx 0,001439$$

$$f(e_6) \approx -0,005096$$

$$\Rightarrow f(x_6) \cdot f(e_6) < 0 \Rightarrow \begin{cases} e_7 \leftarrow e_6 \\ b_7 \leftarrow x_6 \end{cases}$$

$$|e_6 - b_6| = \frac{1}{64} > \varepsilon_2$$

$$\underline{n=7}$$

$$e_7 = e_6 = \frac{39}{64} ; b_7 = x_6 = \frac{79}{128}$$

$$x_7 = \frac{e_7 + b_7}{2} = \frac{157}{256} \approx 0,613281$$

$$f(x_7) \approx -0,001829$$

$$f(e_7) \approx -0,005096$$

$$|e_7 - b_7| = \frac{1}{128} \approx 0,0078 < \varepsilon_2$$

$$[\dots]$$

$$(d) \quad \varepsilon_1 = 0,001 \Rightarrow n \gg \left\lceil \frac{\log |a-b| - \log \varepsilon_1}{\log 2} \right\rceil = \left\lceil 9,97 \right\rceil \Rightarrow \boxed{10}$$

$$\Rightarrow n = 10$$

(e) CONVERGENCE

• BISECTION : MÉTHODE ENCADREMENT

$[a_0, b_0]$ encadre la solution $\Rightarrow$ CONVERGENCE GARANTIE

• NEWTON RAPHSON : CONDITION SUFFISANTE $\rightarrow |F'| < 1$

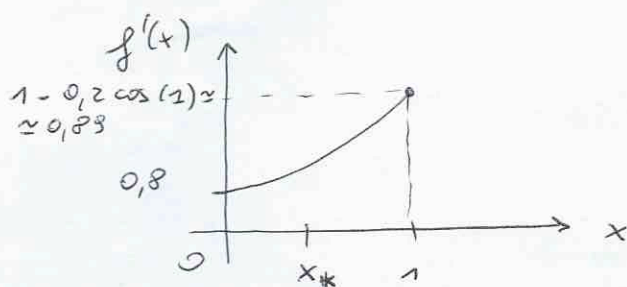
$$F(x) = x - \frac{f(x)}{f'(x)}$$

$$F'(x_*) = 1 - \frac{f'^2 - f f''}{f'^2} \Big|_{x_*}$$

$$\text{si } f'(x_*) \neq 0 \Rightarrow F'(x_*) = 1 - 1 = 0, \text{ car } f(x_*) = 0 \Rightarrow$$

$\Rightarrow$ CONVERGENCE car $|F'(x_*)| < 1$ autour de x_*

Dans notre cas : $f'(x) = 1 - 0,2 \cos x$



$$f'(x) \neq 0 \quad \forall x \in [0, 1]$$

$$x_* \in [0, 1] \Rightarrow f'(x_*) \neq 0 \Rightarrow$$

$\Rightarrow$ CONVERGENCE

$$Ax = \underline{b}$$

$$A = \begin{pmatrix} 8 & -14 & 5 \\ 4 & 3 & -3 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix}$$

• TRIANGULARISATION:

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} 8 & -14 & 5 & 1 \\ 4 & 3 & -3 & 8 \\ 2 & -1 & 0 & 4 \end{array} \right)$$

$$L_2 \rightarrow L_2 - \left(\frac{4}{8} \right) L_1 = L_2 - \frac{1}{2} L_1$$

$\Rightarrow$

$$L_3 \rightarrow L_3 - \left(\frac{2}{8} \right) L_1 = L_3 - \frac{1}{4} L_1$$

$\frac{e_{31}}{e_{11}}$

$$\Rightarrow (A|\underline{b}) = \left(\begin{array}{ccc|c} 8 & -14 & 5 & 1 \\ 0 & 10 & -\frac{11}{2} & \frac{15}{2} \\ 0 & \frac{5}{2} & -\frac{5}{4} & \frac{15}{4} \end{array} \right)$$

$$L_3 \rightarrow L_3 - \left(\frac{5}{2 \cdot 10} \right) L_2 = L_3 - \frac{1}{4} L_2 \Rightarrow$$

$\frac{e_{32}}{e_{22}}$

$$\Rightarrow (A|\underline{b}) = \left(\begin{array}{ccc|c} 8 & -14 & 5 & 1 \\ 0 & 10 & -\frac{11}{2} & \frac{15}{2} \\ 0 & 0 & \frac{1}{8} & \frac{15}{8} \end{array} \right)$$

• SUBSTITUTION EN ARRIÈRE :

6

$$\begin{cases} 8x_1 - 14x_2 + 5x_3 = 1 \\ 10x_2 - \frac{11}{2}x_3 = \frac{15}{2} \\ \frac{1}{8}x_3 = \frac{15}{8} \end{cases}$$

$$\Rightarrow \boxed{x_3 = 15}$$

$$\Rightarrow x_2 = \frac{1}{10} \left(\frac{15}{2} + \frac{11}{2}x_3 \right)_{x_3=15} = \frac{1}{10} \frac{15}{2} (1+11) = \frac{15 \cdot 12}{20} = 9$$

$$\Rightarrow \boxed{x_2 = 9}$$

$$\Rightarrow x_1 = \frac{1}{8} \left(1 + 14x_2 - 5x_3 \right)_{x_2=9, x_3=15} =$$

$$= \frac{1}{8} \left(1 + 14 \cdot 9 - 5 \cdot 15 \right) = \frac{13}{2} = \boxed{\frac{13}{2} = x_1}$$

$$\Rightarrow \underline{x} = \begin{pmatrix} \frac{13}{2} \\ 9 \\ 15 \end{pmatrix}$$

SOLUTION

• VERIFICATION:

$$A \underline{x} = \begin{pmatrix} 8 & -14 & 5 \\ 4 & 3 & -3 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{13}{2} \\ 9 \\ 15 \end{pmatrix} =$$

$$= \begin{pmatrix} 8 \cdot \frac{13}{2} - 14 \cdot 9 + 5 \cdot 15 \\ 4 \cdot \frac{13}{2} + 3 \cdot 9 - 3 \cdot 15 \\ 2 \cdot \frac{13}{2} - 1 \cdot 9 + 0 \cdot 15 \end{pmatrix} = \begin{pmatrix} \frac{2^1}{2^1} \\ \frac{16^8}{2^1} \\ \frac{8^4}{2^1} \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} = \underline{b} \quad (\text{ok!})$$

(a)

$$A = LU$$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ l_2 & l_3 & 1 & 0 \\ l_4 & l_5 & l_6 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ 0 & u_5 & u_6 & u_7 \\ 0 & 0 & u_8 & u_9 \\ 0 & 0 & 0 & u_{10} \end{pmatrix}$$

L1 de A:

$$u_1 = 2 ; u_2 = -1 ; u_3 = 0 ; u_4 = 0$$

L2 de A:

$$l_1 u_1 = -1 \Rightarrow 2l_1 = -1 \Rightarrow l_1 = -\frac{1}{2}$$

$$l_1 u_2 + u_5 = 2 \Rightarrow u_5 = 2 - \frac{1}{2} = \frac{3}{2} = u_5$$

$$l_1 u_3 + u_6 = -1 \Rightarrow u_6 = -1$$

$$l_1 u_4 + u_7 = 0 \Rightarrow u_7 = 0$$

L3 de A:

$$l_2 u_1 = 0 \Rightarrow l_2 = 0$$

$$l_2 u_2 + l_3 u_5 = -1 \Rightarrow l_3 u_5 = -1 \Rightarrow l_3 = -\frac{2}{3}$$

$$l_2 u_3 + l_3 u_6 + u_8 = 2 \Rightarrow l_3 u_6 + u_8 = 2 \Rightarrow u_8 = 2 - \frac{2}{3} = \frac{4}{3} = u_8$$

$$l_2 u_4 + l_3 u_7 + u_9 = -1 \Rightarrow u_9 = -1$$

L_4 de A :

9

$$l_4 v_1 = 0 \Rightarrow l_4 = 0$$

$$l_4 v_2 + l_5 v_5 = 0 \Rightarrow l_5 = 0$$

$$l_4 v_3 + l_5 v_6 + l_6 v_8 = -1 \Rightarrow l_6 v_8 = -1 \Rightarrow l_6 = -\frac{3}{4}$$

$$l_4 v_4 + l_5 v_7 + l_6 v_8 + v_{10} = 2 \Rightarrow v_{10} = 2 - \frac{3}{4} = \frac{5}{4} = v_{10}$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}$$

VERIFICATION

$$A = LU = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$(b) \quad \det(A) = \det(L) \cdot \det(U) = \prod_{i=1}^4 l_{ii} \cdot \prod_{j=1}^4 u_{jj} =$$

$$= \prod_{j=1}^4 u_{jj} = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = \boxed{5 = \det(A)}$$

(c)

10

$$L\underline{z} = \underline{b} \quad ; \quad U\underline{x} = \underline{z} \quad ; \quad \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$L\underline{z} = \underline{b} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

• SUBSTITUTION EN AVANT:

$$\Rightarrow \begin{cases} z_1 = 1 \\ -\frac{1}{2} z_1 + z_2 = 1 \Rightarrow z_2 = 1 + \frac{1}{2} = \frac{3}{2} = z_2 \\ -\frac{2}{3} z_2 + z_3 = 1 \Rightarrow z_3 = 1 + \frac{2}{3} \cdot \frac{3}{2} = 2 = z_3 \\ -\frac{3}{4} z_3 + z_4 = 1 \Rightarrow z_4 = 1 + \frac{3}{4} \cdot 2 = \frac{5}{2} = z_4 \end{cases}$$

$$\Rightarrow \underline{z} = \begin{pmatrix} 1 \\ 3/2 \\ 2 \\ 5/2 \end{pmatrix}$$

$$U\underline{x} = \underline{z} \Leftrightarrow \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \\ 2 \\ \frac{5}{2} \end{pmatrix}$$

• SUBSTITUTION EN ARRIERE:

$$\Rightarrow \begin{cases} \frac{5}{4} x_4 = \frac{5}{2} \Rightarrow x_4 = 2 \\ \frac{4}{3} x_3 - 2 = 2 \Rightarrow x_3 = \frac{4}{3} \cdot \frac{3}{4} = 3 = x_3 \\ \frac{3}{2} x_2 - 3 = \frac{3}{2} \Rightarrow x_2 = \frac{3}{2} \cdot \frac{2}{3} = 3 = x_2 \\ 2x_1 - 3 = 1 \Rightarrow x_1 = 2 \end{cases}$$

$$\Rightarrow \underline{x} = (2, 3, 3, 2)$$

• VERIFICATION:

11

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

A

$$\begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

x

=

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

b

OK!

(a)

$$\ddot{x}(t) + k \sin x(t) = f(t)$$

$$\begin{cases} \ddot{x}(t) = -k \sin x(t) + f(t) \\ x(t_0) = x_0 \\ \dot{x}(t_0) = x_0^{(1)} \end{cases}$$

et $t_0 = 0$ dans cet exercice

$$x_0 = 0$$

$$x_0^{(1)} = 1$$

$$\begin{aligned} x_1 &= x(t) \\ x_2 &= \dot{x}(t) \end{aligned} \Rightarrow \begin{cases} \dot{x}_1 = x_2 = f_1(t, x_1, x_2) \\ \dot{x}_2 = -k \sin x_1 + f(t) = f_2(t, x_1, x_2) \\ x_1(0) = x_0 \\ x_2(0) = x_0^{(1)} \end{cases}$$

$$\vec{x} = (x_1, x_2)$$

$$\vec{F}(t, \vec{x}) = (f_1, f_2) = (x_2, -k \sin x_1 + f)$$

$$\Rightarrow \vec{x}_{n+1} = \vec{x}_n + h \vec{F}(t_n, \vec{x}_n)$$

EULER EXPLICITEavec h : PAS de DISCRETISATION
(PAS de TEMPS)

(b)

$$g(t) = \sin(3t)$$

$$k=2$$

$$x(0) = 0$$

$$\dot{x}(0) = 1$$

$$h = 0,15$$

$$f_1(0, x_1(0), x_2(0))$$

$t=h$:

$$\begin{cases} x_1(h) = x_1(0) + h \cdot x_2(0) \\ x_2(h) = x_2(0) + h \cdot \underbrace{\{-k \sin[x_1(0)] + g(0)\}}_{f_2(0, x_1(0), x_2(0))} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1(h) = 0 + h \cdot 1 = 0,15 \\ x_2(h) = 1 + h \cdot \{-k \sin(0) + \sin(3 \cdot 0)\} = 1 \end{cases}$$

$t=2h$

$$x_1(2h) = x_1(h) + h \cdot x_2(h) = 0,15 + 0,15 \cdot 1 = 0,3$$

$$x_2(2h) = x_2(h) + h \cdot \{-k \sin(x_1(h)) + g(h)\} =$$

$$= 1 + 0,15 \cdot \left\{ -2 \cdot \underbrace{\sin(0,15)}_{\approx 0,15} + \underbrace{\sin(3 \cdot 0,15)}_{\approx 0,45} \right\} =$$

$$\approx 1 + 0,15 \cdot \{-2 \cdot 0,15 + 0,45\} =$$

$$= 1 + 0,15 \cdot 0,15 = 1,0225$$

$\Rightarrow$

$$x(h) = x_1(h) = 0,15$$

$$x(2h) = x_1(2h) = 0,3$$