Ex 1 a) $\frac{d}{dt} H(t) = \underbrace{-\frac{d^2}{D^2}\sqrt{2g\,H(t)}}_{F(t, H(t))}$

$$J = \frac{\partial F}{\partial H} = \frac{\partial}{\partial H}\left(-\frac{d^2}{D^2}\sqrt{2gH}\right) = \underbrace{-\frac{d^2}{D^2}\sqrt{2g}}_{<0} H^{-3/2}$$

$H(t) > 0 \Rightarrow J < 0 \quad \forall H$ TOUJOURS STABLE

b) RK2 $y_{k+1} = y_k + h f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2} f(x_k, y_k)\right)$

$$f(x_k, y_k) = -\frac{d^2}{D^2}\sqrt{2g\,H_k}$$

$$f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2} f(x_k, y_k)\right) =$$

$$= -\frac{d^2}{D^2}\sqrt{2g\left(H_k + \frac{h}{2}\left(-\frac{d^2}{D^2}\sqrt{2g\,H_k}\right)\right)}$$

$$H_{k+1} = H_k - h\frac{d^2}{D^2}\sqrt{2g\,H_k - \frac{d^2}{D^2}\frac{h}{2}(2g)^{3/2}\sqrt{H_k}}$$

c) $h = \frac{1}{2}s \quad D = 4m \quad d = 0{,}4\ m \quad g \sim 10\ m/s^2$

$$H_{k+1} = H_k - \frac{1}{2}\left(\frac{0.4}{4}\right)^2\sqrt{20\,H_k - \left(\frac{0.4}{4}\right)^2\frac{1}{4}(20)^{3/2}\sqrt{H_k}}$$

$$H_{k+1} = H_k - \frac{1}{200}\sqrt{20\,H_k - \sqrt{\frac{H_k}{20}}}$$

$h=0 \quad H_0 = 10\ m$

$h=0.5 \quad H_1 = 10 - \frac{1}{200}\sqrt{20\cdot 10 - \sqrt{\frac{10}{20}}} = 10 - \frac{1}{200}\sqrt{200 - \sqrt{\frac{1}{2}}} =$

$\simeq 9.93\ m$

$h=1 \text{ sec} \quad H_2 = 9.93 - \frac{1}{200}\sqrt{20\cdot 9.93 - \sqrt{\frac{9.93}{20}}} \simeq 9.86\ m$

$h = 1.5\ sec \quad H_3 = 9.79\ m$

Ex 2

a)

$$A = L \cdot U$$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & l_3 & 1 \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \\ 0 & u_4 & u_5 \\ 0 & 0 & u_6 \end{pmatrix}$$

$4 = u_1 \qquad -2 = u_2 \qquad 0 = u_3$

$-2 = l_1 u_1 \qquad 2 = l_1 u_2 + u_4 \qquad -1 = l_1 u_3 + u_5$

$0 = l_2 u_1 \qquad -1 = l_2 u_2 + l_3 u_4 \qquad 2 = l_2 u_3 + l_3 u_5 + u_6$

$\boxed{u_1 = 4} \quad \boxed{u_2 = -2} \quad \boxed{u_3 = 0}$

$\boxed{l_1 = -\frac{2}{4} = -\frac{1}{2}}$ $\qquad 2 = -\frac{1}{2}(-2) + u_4 \Rightarrow \boxed{u_4 = 1}$, $\qquad -1 = -\frac{1}{2} \cdot 0 + u_5 \Rightarrow \boxed{u_5 = -1}$

$\boxed{l_2 = 0} \qquad -1 = 0 + l_3 \Rightarrow \boxed{l_3 = -1}$

$2 = -1(-1) + u_6 \Rightarrow \boxed{u_6 = 1}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow L\underbrace{U\vec{x}}_{\vec{z}} = \vec{b}$$

$$\begin{cases} L\vec{z} = \vec{b} \\ U\vec{x} = \vec{z} \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \rightarrow \quad z_1 = 2$$

$$-\frac{1}{2} z_1 + z_2 = 6 \rightarrow z_2 = 7$$

$$-z_2 + z_3 = 1 \rightarrow z_3 = 8$$

$$\begin{pmatrix} 4 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix} \rightarrow \quad x_3 = 8$$

$$x_2 - x_3 = 7 \rightarrow x_2 = 15$$

$$4x_1 - 2x_2 = 2 \rightarrow x_1 = 8$$

$$\vec{x} = \begin{pmatrix} 8 \\ 15 \\ 8 \end{pmatrix}$$

b) OUI CAR LA MATRICE EST SYMETRIQUE ET DEFINIE POSITIVE

- MATRICE SYMETRIQUE $A = A^T$
- MATRICE DEFINIE POSITIVE $\vec{x}^T A \vec{x} > 0 \quad \forall x$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (a,b,c)\begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} =$$

$$= (a,b,c)\begin{pmatrix} 4a-2b \\ -2a+2b-c \\ -b+2c \end{pmatrix} =$$

$$= (4a-2b)a + b(-2a+2b-c) + c(-b+2c) =$$
$$= 4a^2 - 2ab - 2ab + 2b^2 - bc - bc + 2c^2 =$$
$$= 4a^2 - 4ab + 2b^2 - 2bc + 2c^2 =$$
$$= (4a^2 - 4ab + b^2) + (b^2 - 2bc + c^2) + c^2 =$$
$$= (2a-b)^2 + (b-c)^2 + c^2 > 0$$
$$\forall a,b,c$$

c) $$\vec{x}_{(k+1)} = \Pi_J \vec{x}_{(k)} + \vec{c}_J$$

$$\Pi_J = D^{-1}(D-A) \qquad \vec{c} = D^{-1} b$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Pi = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\left[\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\vec{c} = D^{-1} b = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ \frac{1}{2} \end{pmatrix}$$