Rayleigh-Bénard convection using Lattice Boltzmann method

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1 Introduction

This work is devoted to the study of the Rayleigh Bénard system using Lattice Boltzmann method of simulation with Boussinesq approximation. Using this model, the study investigates the 2D Rayleigh Bénard problem to determine the critical Rayleigh number for a Newtonian fluid. This is determined by two techniques. The first technique is based on the plot between Nusselt number and Rayleigh number whereas the second technique is based on the plot involving the logarithm of the average horizontal velocity squared. In this study, a Poiseuille flow is also examined to determine the effect on the maximum velocity with the change in Pressure gradient. A combined effect of Poiseuille flow and buoyancy effects is used to study the relation between the critical Rayleigh number and the Reynold's number for a Newtonian fluid. Lastly, using the Carreau-Yasuda model the phenomenon is simulated for a Non-Newtonian fluid and the variation between the critical Rayleigh number and Reynold's number is studied for a shear thickening and a shear thinning fluid.

2 Formulation

Rayleigh Bénard convection is a fundamental phenomenon found in many atmospheric and industrial applications. It is a type of natural convection occurring in a plane of fluid heated from below in which the fluid develops a regular pattern of convection cells known as Bénard cells. Due to the heating of the fluid the density at the bottom becomes lighter than at the top. If the fluid is heated sufficiently large enough, then the top state becomes unstable and convective motion occurs. The primary instability represents a transition from diffusive thermal conduction to stationary time independent steady convection at a critical Rayleigh number.

The formulation for Rayleigh Bénard convection using Lattice Boltzmann method is accomplished by establishing 2 distribution functions for the flow and temperature fields. The distribution functions f and g are defined as probability of particles at site x at time t moving with particle velocity c_i during the interval Δt in each lattice direction i. The two distribution functions follow the Lattice Boltzmann transport equations with the single relaxation Bhatnagar-Gross-Krook (BGK) approximation i.e.

$$f_i(x + c_i\Delta t, t + \Delta t) - f_i(x, t) = \frac{\Delta t[f_i^{eq}(x, t) - f_i(x, t)]}{\tau_\nu} + \dot{J}_i \tag{2.1}$$

for the flow field

$$g_i(x + c_i\Delta t, t + \Delta t) - g_i(x, t) = \frac{\Delta t[g_i^{eq}(x, t) - g_i(x, t)]}{\tau_D} \tag{2.2}$$

for the temperature field

Where $\dot{J}_i$ is the momentum input from the buoyant body force, τ_ν and τ_D are the relaxation times for flow and temperature LB equations respectively.

The kinematic viscosity ν is given by $\nu = c_s^2(\tau_\nu - 0.5)$ and the thermal diffusivity κ is given by $\kappa = c_s^2(\tau_D - 0.5)$. where $c_s = c/\sqrt{3}$ is the speed of sound.

The flow properties are defined by :

Flow density:

$$\rho = \sum_i f_i \tag{2.3}$$

Momentum flux:

$$\rho u_A = \sum_{iA} f_i c_{iA} \tag{2.4}$$

Temperature:

$$\theta = \sum_i g_i \tag{2.5}$$

Using the Chapman-Enskog expansion, the continuity equation and the Navier-Stokes equations can be recovered exactly at the second-order approximation from the Lattice Boltzmann equation of the flow field. Similarly, the convective-diffusion equation can be obtained from the Lattice Boltzmann equation of the temperature field.

The boundary conditions are implemented using the bounce back condition for the Lattice Boltzmann method along with the no slip condition on the walls. The viscous heat dissipation is neglected for the incompressible flow to allow the use of the Lattice Boltzmann method.

The expression used for determining the Nusselt number is as follows:

$$Nu = \frac{\langle v_y T\rangle - \kappa\partial_y\langle T\rangle}{\kappa\Delta T/H} \tag{2.6}$$

The Rayleigh number is calculated by the following expression:

$$Ra = \frac{g\beta\Delta T H^3}{\nu\kappa} \tag{2.7}$$

where g is the acceleration due to gravity , β is the thermal expansion coefficient and H is the distance between the plates.

In this study, Poiseuille flow is also examined which is used as a benchmark to check the accuracy of the model. The planar Poiseuille flow is the steady flow between infinite parallel plates. The symmetries of the numerical geometry reduce the incompressible Navier-Stokes equation to

$$-\partial_x p + \partial_y(\nu\partial_y u_x) = 0 \tag{2.8}$$

Integrating this equation one gets

$$-\partial_x p y + C_1 + \nu\partial_y u_x = 0 \tag{2.9}$$

By symmetry we have

$$C_1 = \partial_x p\frac{H}{2} \tag{2.10}$$

The velocity in the horizontal direction is a function of y. The expression for the velocity is as follows:

$$u_x(y) = \frac{\nabla P(y(Ny) - y^2)}{2\nu} \tag{2.11}$$

By differentiating the above equation the expression for maximum velocity is as follows:

$$u_{max} = \frac{\nabla P(Ny)^2}{8\nu} \tag{2.12}$$

The non Newtonian part was simulated by using the Carreau-Yasuda model. This model is a modification of the power law. The power law is as follows:

$$\mu_s(\dot{\gamma}) = k\dot{\gamma}^{n-1} \tag{2.13}$$

where n is the non-dimensional power-law index , k is a constant , μ is the viscosity and $\dot{\gamma}$ is the shear rate. In the limit $n = 1$ we recover the newtonian case with $\mu_s = k$. The cases $n < 1$ and $n > 1$ represent respectively the shear thinning and shear thickening fluids. However, the power law fails to represent real fluids in the limits of small and high shear rate. The Carreau-Yasuda model accounts for these flaws and is expressed as follows:

$$\frac{\mu_s - \mu_\infty}{\mu_0 - \mu_\infty} = (1 + (\lambda\dot{\gamma})^a)^{(n-1)/a} \tag{2.14}$$

where λ is a time constant, a is a dimensionless number, μ_0 and μ_∞ are the viscosities at zero and infinite shear rate.

3 Results and Discussion

The simple thermal Lattice Boltzmann model was implemented by considering natural convection between the two plates by maintaining a temperature difference of 0.1. The gravity in non dimensional terms was taken as 0.001. The relaxation times were taken as unity. The Prandtl number for the simulations was also maintained constant at unity. The simulations were performed by varying the value of the thermal expansion coefficient thus changing the Rayleigh number.

The plot between Nusselt number and time for a Rayleigh number of 2250 is shown in figure 1. It shows a stable value upto the primary instability which represents a transition from diffusive thermal conduction to stationary time independent steady convection at the critical Rayleigh number. The velocity profile for fully developed Rayleigh Bénard cells is shown in figure 2. The temperature profiles before and after the development of the Rayleigh Bénard cells are shown in Figure 3 and 4 respectively.

The value of the critical Rayleigh Number was found out for a Newtonian fluid by measuring the steady state value of the Nusselt number. The value of the thermal expansion coefficient β was varied from 0.05 to 10 thus varying the Rayleigh number from 225 to 45000. At each value of β the simulation was made to run for a time step equal to 200000 and the steady state value of the Nusselt number was recorded.The critical Rayleigh number obtained by this method was 1845. However, since the simulation would run for only a fixed time step this method would not lead to an accurate value of the critical Rayleigh number as the more we reach closer to the critical value the greater time it would take for development of the Bénard

cells. An approximate emperical relation between the Nusselt number and Rayleigh number is as follows :

$$Nu = 1.56(\frac{Ra}{Ra_c})^{0.296} \tag{3.1}$$

A comparison between the plot obtained between Nu and Ra and the emperical relation is shown in Figure 5.

The second technique based on the average horizontal velocity was a more accurate technique. It involved finding out the slope of the graph between logarithm of average horizontal velocity squared with time. It was observed that the critical Rayleigh number value is obtained when the slope of this graph is zero. The value of slope is measured for 2 different resolutions. First by keeping the elements in the X direction equal to 101 and in the Y direction equal to 50 and then doubling the value. The critical values obtained were 1720.35 and 1710.9 respectively. It was also observed that doubling the value resulted in a more accurate value of the critical Rayleigh number. A major advantage of this method was that only by measuring the slope for a short period it could be determined whether the Rayleigh number is greater or less than the critical value. Thus, reducing the overall time for simulation. The graph obtained between the slope and Rayleigh number by this method is shown in Figure 6.

The simulation based on the Poiseuille flow was implemented to check the variation in the velocity with a change in the pressure gradient.The maximum velocity was found out by fitting a parabola to the velocity profile. The coefficients of the parabola were then used to determine the maximum value of the velocity. A linear plot was obtained between the pressure gradient and the maximum velocity. The plot between the maximum horizontal velocity and the pressure gradient is shown in Figure 7.

The simulations involving combined Poiseuille flow and buoyancy effects for a Newtonian fluid was accomplished by simultaneously implementing the Lattice Boltzmann code for the Rayleigh Bénard simulation and the code for Poiseuille flow. This was used to study the relation between the Reynold's number and the critical Rayleigh number. An approximate procedure was adopted for finding out this relation. The value of the Nusselt number was recorded at a time step of 100000 for a particular Rayleigh number. This was then done for other values of Rayleigh number and the critical Rayleigh number was determined when the value of Nusselt number showed a deviation of the order of 10^{-3} from the initial steady value of 1. The Reynold's number was obtained by using the maximum velocity. The expression used for determining the Reynold's number is as follows:

$$Re = \frac{U_{max}(Ny)}{\nu} \tag{3.2}$$

where $\nu = (1 - 0.5)/3 = 1/6$ is the kinematic viscosity in non dimensional terms.

Using this method an approximate plot between the critical Rayleigh number and the Reynold's number was obtained and is shown in Figure 8.

The simulation for the non Newtonian fluid is accomplished by simultaneously implementing the non-Newtonian model, the Rayleigh Bénard simulation and the code for Poiseuille flow. The simulation was done for values of n=0.1(shear thinning) and 2(shear thickening) and the plot between the critical Rayleigh number and the Reynold's number was obtained. The viscosity at infinite shear is maintained at $1/5^{th}$ of the viscosity at zero shear. The graph between the critical Rayleigh number and Reynold's number is shown in Figure 9.

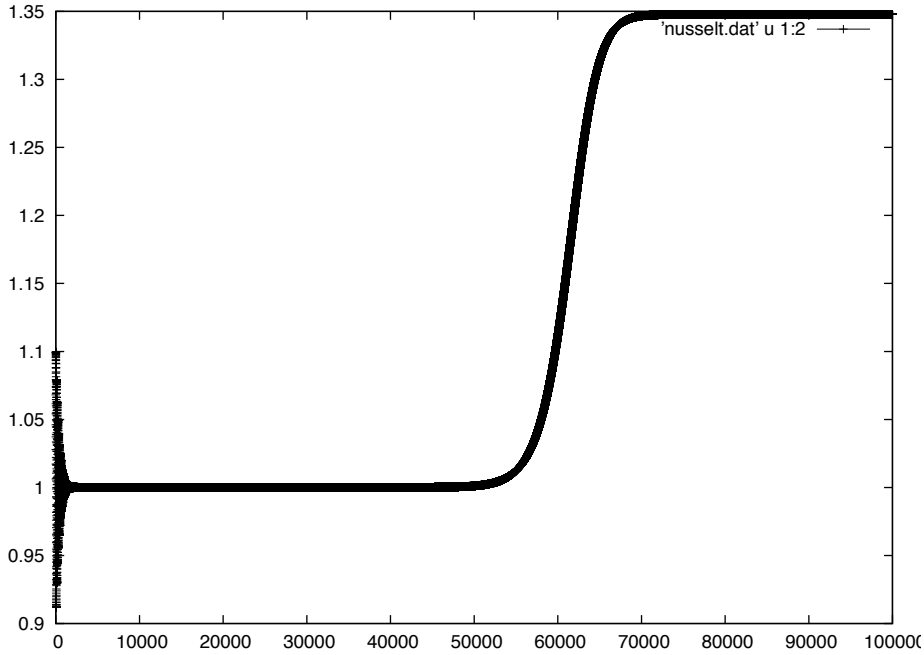

Figure 1: Nusselt number versus time for Ra=2250

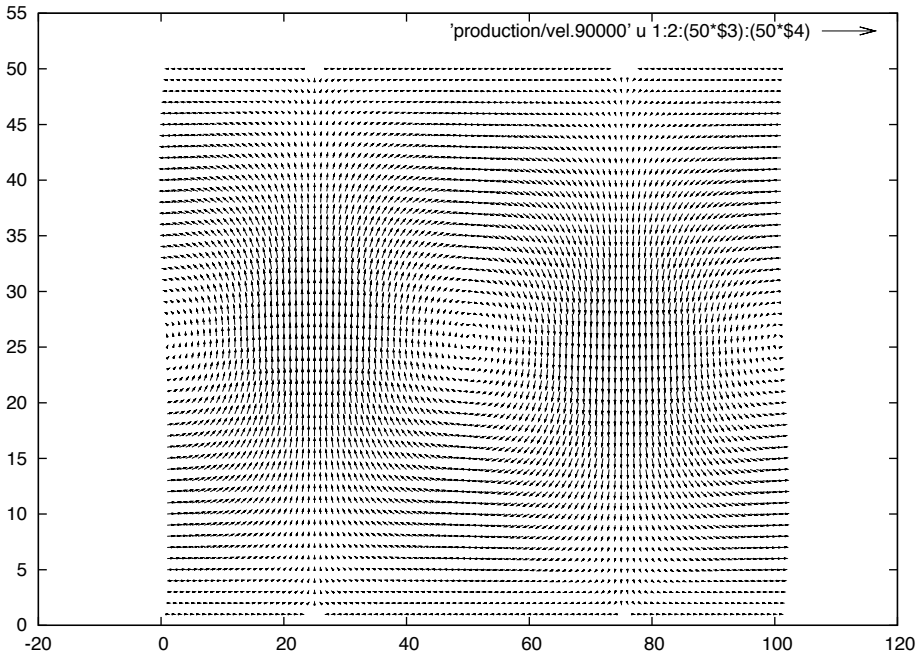

Figure 2: Velocity profile at time step 90000 at Ra=2250

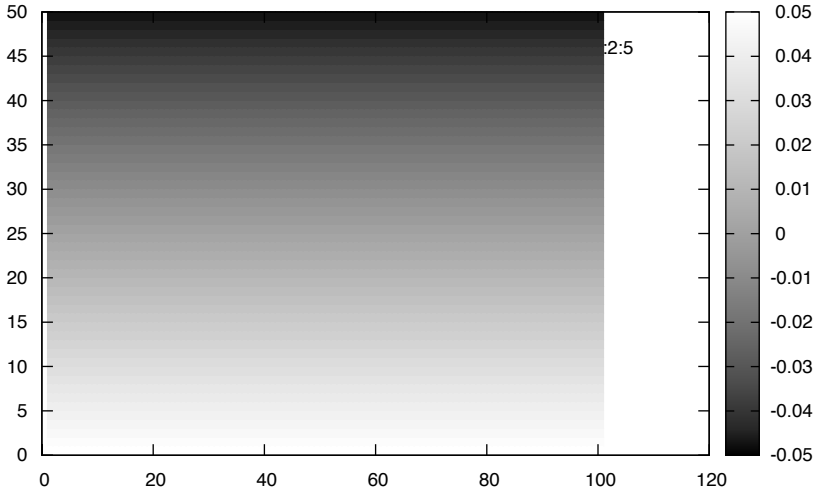

Figure 3: Temperature profile at time step 20000 at Ra=2250

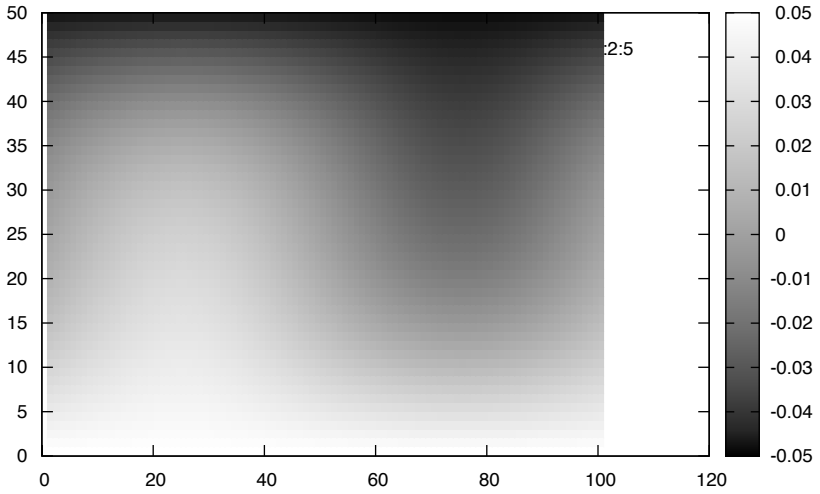

Figure 4: Temperature profile at time step 90000 at Ra=2250

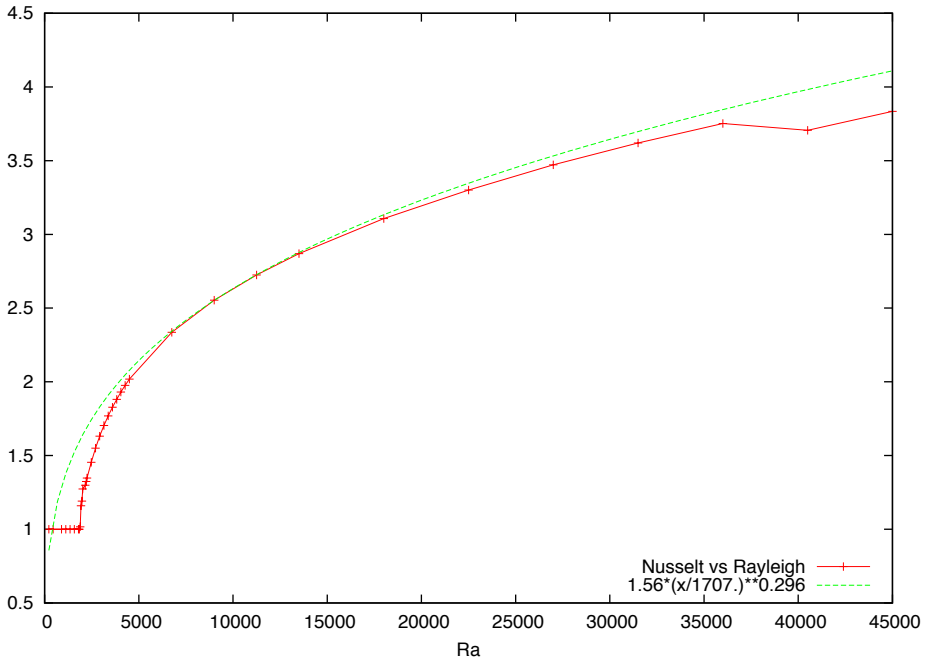

Figure 5: Nusselt number versus Rayleigh number obtained by determining the steady state value of Nusselt number for different Rayleigh numbers

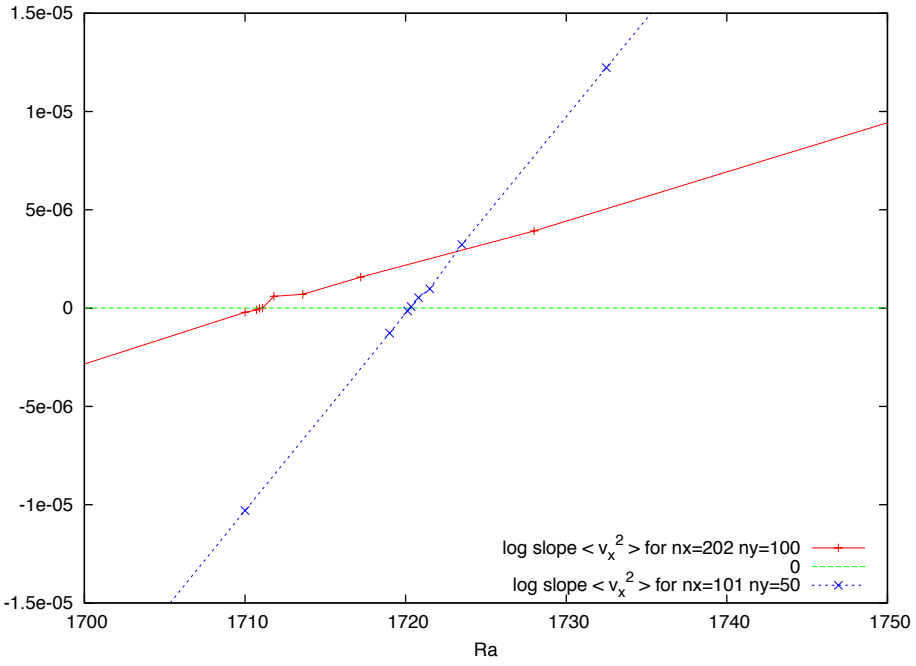

Figure 6: slope of logarithm of average horizontal velocity squared with time versus Rayleigh number

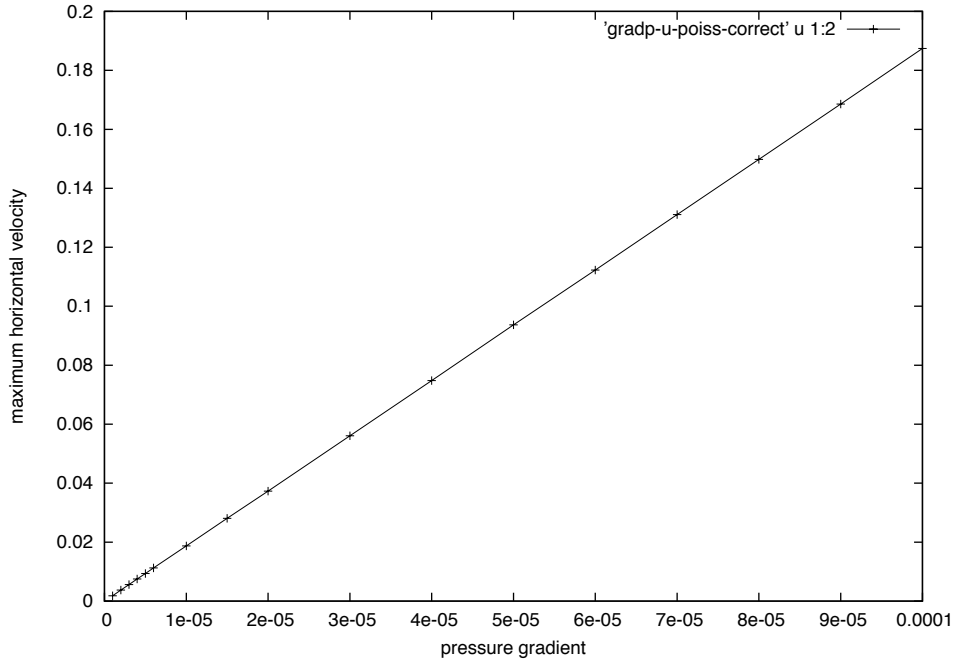

Figure 7: Maximum horizontal velocity versus pressure gradient for a Poiseuille flow

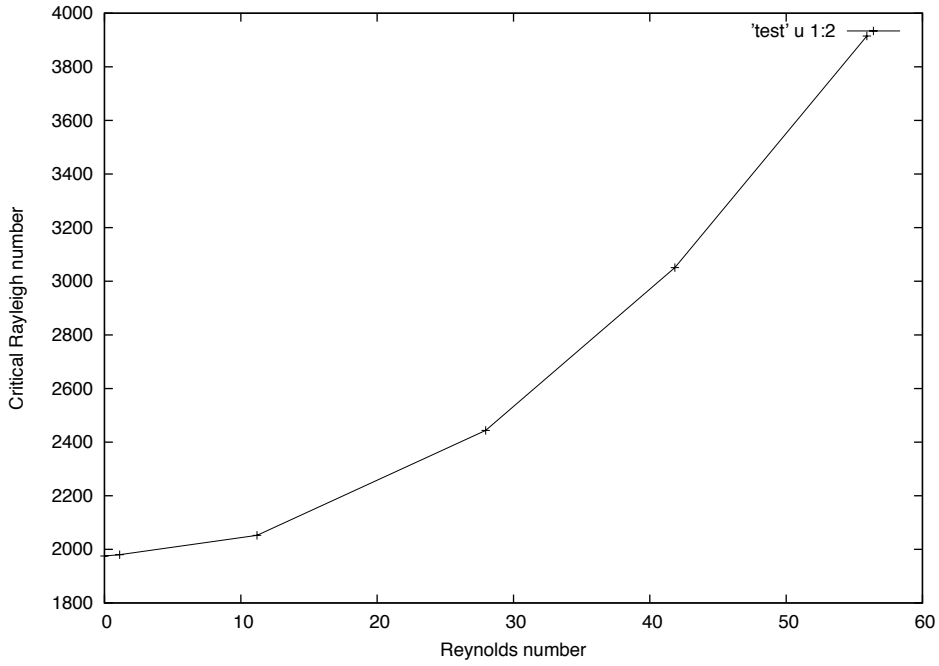

Figure 8: Critical Rayleigh number versus Reynold's number for a Newtonian fluid

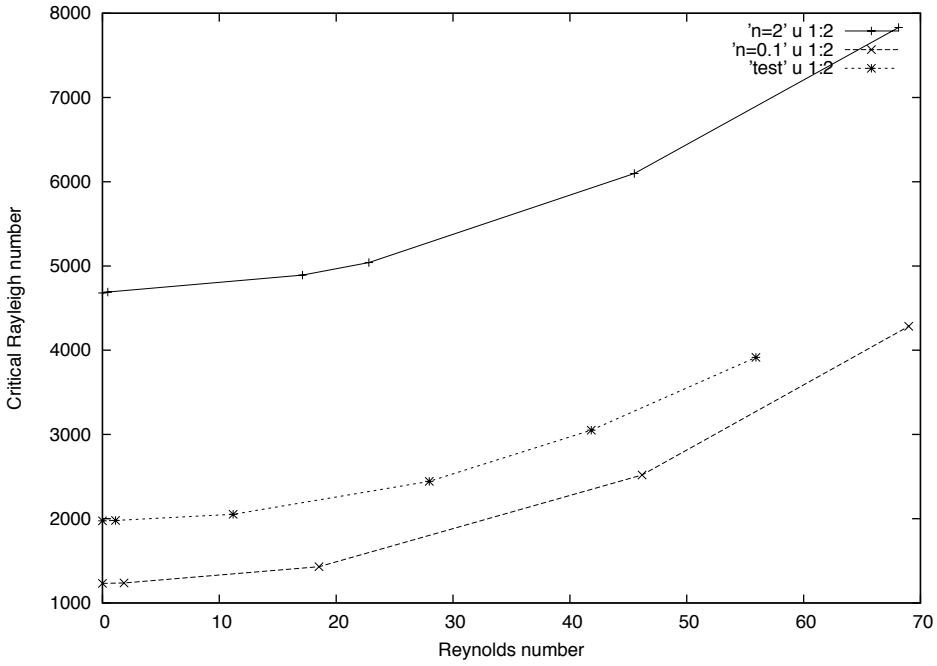

Figure 9: Critical Rayleigh number versus Reynold's number for a Newtonian and non Newtonian fluid