

# Evidences of Bolgiano-Obhukhov scaling in three-dimensional Rayleigh-Bénard convection

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We present different results from high-resolution high-statistics direct numerical simulations of a three-dimensional convective cell. We test the fundamental physical picture of the presence of both a Bolgiano-Obhukhov-like and a Kolmogorov-like regime. We find that the dimensional predictions for these two distinct regimes (characterized, respectively, by an active and passive role of the temperature field) are consistent with our analysis.

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Dimensional hypothesis for homogeneous and isotropic turbulence have been formulated in the works of Kolmogorov [1] long time ago. On the other hand, a clear theoretical picture is still missing for the strong fluctuations in the energy dissipation field that lead to intermittency effects (i.e., non-Gaussian behavior of probability distribution functions). Phenomenological theories have been proposed [1] but no systematic theory for computing experimentally measured numbers has been successful so far. The situation of “non-ideal” turbulence is even more controversial, already at the level of dimensional expectations. A typical realization, the one we will address in this paper, is the three-dimensional (3D) Rayleigh-Bénard cell, described, in the Boussinesq approximation [2], by the following set of equations:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \alpha g T \hat{z}, \quad (1)$$

$$\partial_t T + (\mathbf{v} \cdot \nabla) T = \chi \nabla^2 T \quad (2)$$

with isothermal boundary conditions on the upper and lower planes of a cell of height  $H$ :  $T(z=0) = +\Delta T/2$  and  $T(z=H) = -\Delta T/2$ . As usual,  $\mathbf{v}(\mathbf{x}, t)$  is the velocity field and  $T(\mathbf{x}, t)$  the temperature field. Kinematic viscosity and thermal diffusivity are, respectively,  $\nu$  and  $\chi$ , while the thermal expansion coefficient is  $\alpha$  and gravity acceleration is  $g$ . In the following we will mainly focus on the longitudinal structure functions of  $v$  and  $T$ :  $S_p(r) = \langle \{[\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \hat{\mathbf{r}}\}^p \rangle$  and  $T_p(r) = \langle [T(\mathbf{x}+\mathbf{r}) - T(\mathbf{x})]^p \rangle$ .

In this work we present some tests of the predictions for the structure functions defined above that can be derived in the scenario proposed years ago by Bolgiano and Obhukhov [3] to describe convective turbulence.

Despite much research on the subject [4], sound evidence of the validity of the Bolgiano-Obhukhov scenario and the recovery of Kolmogorov scaling at small scales is still missing. Good quality confirmation of the dimensional Bolgiano-Obhukhov scenario was recently shown in two-dimensional numerical simulations [5]. This result cannot be directly related to the 3D case because of the strong differences in the properties of the velocity field in 2D. Statistical properties of the velocity field were recently used with even simpler models (shell models for turbulence) [6]. This approach is even less probing since these models were built precisely in order

to implement Bolgiano-Obhukhov scaling. Finally, a recent experiment even questioned the behavior of the Bolgiano length inside a convective cell [7]. Of course from an experimental point of view it can be difficult, if possible at all, to measure all relevant quantities and it can be even harder to have access to information at several (not just a few) positions inside the volume. The results presented in this paper show good consistency with the idea of a Bolgiano-Obhukhov regime (i.e., a range of length scales where temperature driven buoyancy effects are dominant) but, because of the still limited resolution of “state of the art” numerical simulations, we will have to resort to somehow indirect tests.

This paper is organized as follows: a brief review of phenomenological expectations, details of our numerical simulations, data analysis and then concluding remarks.

Starting from Eqs. (1) and (2), if one uses dimensional analysis and assumes homogeneous scaling for velocity and temperature differences (inside the inertial range), one ends up with two distinct scaling regimes. At small scales (Kolmogorov-like scenario),  $r \ll L_B$ ,

$$\delta v(r) \sim \varepsilon^{1/3} r^{1/3}, \quad (3)$$

$$\delta T(r) \sim N^{1/2} \varepsilon^{-1/6} r^{1/3}, \quad (4)$$

while at large scales (Bolgiano-Obhukhov-like scenario),  $r \gg L_B$ ,

$$\delta v(r) \sim (\alpha g)^{2/5} N^{1/5} r^{3/5}, \quad (5)$$

$$\delta T(r) \sim (\alpha g)^{-1/5} N^{2/5} r^{1/5}. \quad (6)$$

The Bolgiano length  $L_B$ , is an estimate of the distance at which the dissipative and buoyancy terms on the right hand side of Eq. (1) balance, valid under the assumption of a scaling behavior. In the following we will use the  $z$ -dependent version of  $L_B(z)$  introduced in Ref. [8] that uses averages of  $\varepsilon$  and  $N$  defined at a given height  $z$ ,  $\varepsilon(z) = (\nu/2) \langle \sum_{ij} (\partial_i v_j)^2 \rangle_z$  and  $N(z) = (\chi/2) \langle \sum_i (\partial_i T)^2 \rangle_z$ :

$$L_B(z) = \frac{\varepsilon(z)^{5/4}}{N(z)^{3/4} (\alpha g)^{3/2}}. \quad (7)$$

Our analysis has been performed on data coming from direct numerical simulations (DNS) employing a standard

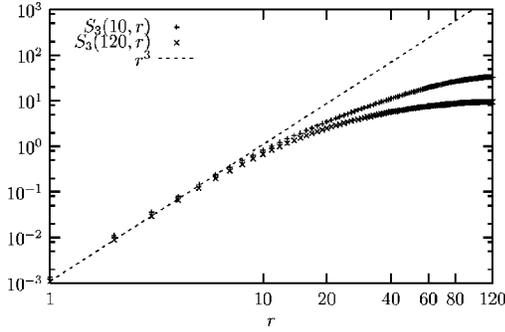


FIG. 1. Log-log plot of the structure function  $S_3(r)$ , defined in the text, measured close to the end plates (+) and at the center of the cell ( $\times$ ). The horizontal scale is in grid points, while the vertical scale is in arbitrary units.

lattice Boltzmann scheme [9,14], on a massively parallel computer [10]. The resolution of the numerical simulation was  $240^3$  and the Rayleigh number [ $Ra = \alpha g \Delta T H^3 / (\nu \chi)$ ] was approximately  $3.5 \times 10^7$ . The Prandtl number was equal to unity and its precise value not relevant for our analysis. We performed a stationary simulation extending over approximately 500 recirculation times and stored nearly 400 independent configurations with complete information on all velocity components and the temperature field. Boundary conditions were periodic in the  $x$  and  $y$  directions (in order to maintain homogeneity on horizontal planes) and isothermal at the top and bottom planes of the cell ( $z=0$  and  $z=H$ ).

Stress-free boundary conditions (i.e., free slip) were used on the top/bottom planes for the velocity field. This choice was made in an attempt to reduce the effects of a viscous boundary layer close to the horizontal walls. Indeed, as it will be clear from the following, thermal effects are dominant near the isothermal walls, so that using no-slip boundary conditions might produce effects on the velocity statistics interfering with the ones coming from pure buoyancy. More details on this point will be given later on.

The most direct way to test the dimensional validity of the Bolgiano-Obukhov picture would be to measure structure functions and to check whether they scale with exponents close (apart from intermittency corrections) to the ones predicted by the sets of equations (3),(4) and (5),(6).

Unfortunately this cannot be done directly because of the limited resolution of our DNS.

Figure 1 substantiates this comment, by plotting the velocity structure function of order 3,  $S_3(z, r) = \langle [v(x+r, z) - v(x, z)]^3 \rangle$  at two different positions: one very close to the wall ( $z=10$ ), and the other at the center of the cell ( $z=120=H/2$ ).

As it can be seen from Fig. 1, no evident scaling range can be detected, even if a steepening of  $S_3(r)$  is clearly detected as one comes close to the walls.

Given this state of affairs, in order to test the consistency with the Bolgiano-Obukhov picture, we performed two distinct, albeit less direct, tests. The first test consists in checking that the Bolgiano length actually keeps track of the scales at which the buoyancy term balances the dissipative term in Eq. (1).

First, we measure  $L_B(z)$  in terms of  $\varepsilon(z)$  and  $N(z)$  and

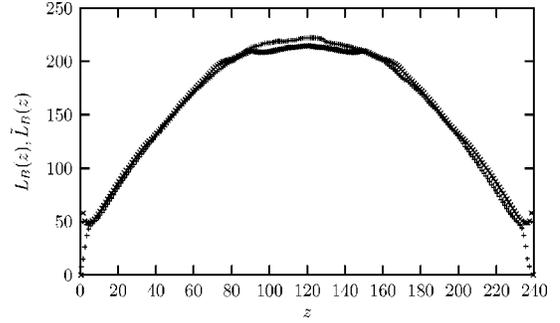


FIG. 2. Plot of the  $z$ -dependent Bolgiano length as a function of the distance from the bottom wall  $z$  as given by the usual definition of Eq. (7) [ $L_B(z)$ , +] and as given by the procedure described in the text, following Eq. 8 [ $\alpha \tilde{L}_B(z)$ ,  $\times$ ]. All dimensions are in grid points and  $\alpha=3.1$ .

plot our results in Fig. 2. From a first look at the behavior of  $L_B(z)$  we learn that Bolgiano effects should be measurable, if at all, near to the isothermal walls [where  $L_B(z)$  is of the order of  $10^1$ ].

Close to the center of the cell,  $L_B$  is of the order of  $10^2$  and Kolmogorov-like behavior is expected to be measurable at almost all scales.

We then measure, directly and independently of the previous quantity, the scale at which dissipation and buoyancy effects balance, i.e., we look for the scale  $\tilde{L}_B(z)$  such that

$$\varepsilon(z) \sim \alpha g \langle \delta v(\tilde{L}_B) \delta T(\tilde{L}_B) \rangle_z. \quad (8)$$

We provisionally consider  $\tilde{L}_B$  a modified definition of the Bolgiano length. In Fig. 2 we plot  $L_B(z)$  and  $\alpha \tilde{L}_B(z)$  with a constant which was tuned to be  $\alpha \sim 3.1$ .

As it can be seen, the two definitions yield the same behavior apart from the multiplicative factor  $\alpha$ . The reason for the multiplicative factor (of order unity) is due to the fact that the two definitions are dimensional estimates so they can miss a numerical prefactor. Considering this, the fact that the two definitions behave in the same way after rescaling has to be regarded an excellent agreement.

Here we want to underline two points which, we believe, add relevance to this finding. First of all the two definitions are of course linked but definitely different. The “traditional” definition of  $L_B$  (see also [8]), as from definition (7), comes from supposing that the two scaling laws in Eq. (3) and (5) merge at  $L_B$ , hence from solving the equation  $\varepsilon(z)^{1/3} L_B(z)^{1/3} = (\alpha g)^{2/5} N(z)^{1/5} L_B(z)^{3/5}$ . The second definition,  $\tilde{L}_B(z)$  comes instead from a *direct* measurement of the strength of the dissipative and forcing term in Eq. (1).

The second important point consist in the fact that our cell is *not* homogeneous; this adds strength to the equality between the two different definitions of  $L_B(z)$ .

As a consequence of this test, we can claim that the Bolgiano-Obukhov scenario and the expected power law behavior are consistent with the value measured for the terms appearing on the right hand side of Eq. (1). Furthermore we fully confirm (with higher accuracy) our former results for  $L_B$  [8]. We like to underline that extracting the behavior of

the Bolgiano length using measured quantities that are not those appearing in its definition could introduce large errors. For example, in Ref. [7] a Bolgiano length was extrapolated as the scale at which there is a change of slope in a particular structure function. In using such a procedure one is dominated by strong finite size effects present on the structure functions.

We now proceed to our second test. Under the hypothesis of validity of Bolgiano-Obhukhov scaling, and if enough resolution were available, one would expect to see power law behavior in the inertial range in Fig. 1 with two distinct slopes 1 and  $9/5$ , corresponding to Kolmogorov and Bolgiano-Obhukhov scaling, respectively. With available resolution, we are not able to detect a clear power law behavior from Fig. 1, although we see a clear steepening of the structure function as we come close to the wall. What we are going to do in the following is to try to quantify as well as possible this change of slope.

We adopt the following procedure. We focus on the plot of  $S_6(r)$  vs  $S_3(r)$  and apply extended self similarity (ESS, see [11]) in order to detect a trustable plateau in the local slopes. We found this plateau to correspond roughly to distances in the interval  $\mathcal{I}^v = [25, 40]$  for the velocity structure function and  $\mathcal{I}^T = [15, 30]$  for the structure functions of the temperature field. We then define two other intervals slightly shifted to the left  $\mathcal{I}^v_- = [20, 35]$ ,  $\mathcal{I}^T_- = [10, 25]$  and to the right  $\mathcal{I}^v_+ = [30, 45]$ ,  $\mathcal{I}^T_+ = [20, 40]$ . These intervals were shifted by a reasonable amount, i.e., were still possible highest or lowest estimate for the same plateau.

We finally perform a power law fit to extract a scaling exponent on the structure functions for the velocity and for the temperature in the three intervals previously defined.

We define the scaling exponents for the structure functions of interest as follows:

$$\langle \delta v(z, r)^p \rangle \sim r^{\zeta_p(z)}, \quad (9)$$

$$\langle \delta T(z, r)^p \rangle \sim r^{\chi_p(z)}, \quad (10)$$

$$\langle [\delta v(z, r) \delta T(z, r)^2]^{p/3} \rangle \sim r^{\rho_p(z)}. \quad (11)$$

Our fits provide a central value for the exponents and two, respectively, higher and lower estimates (corresponding to the shifted intervals). The procedure adopted in estimating the errors reflects the fact that the largest source of systematic error is connected with the choice of the fitting ranges and not with statistical accuracy.

In the Kolmogorov regime we expect the exponents of Eqs. (9)–(11) to take the following dimensional values:  $\zeta_p = p/3$ ,  $\chi_p = p/3$ , and  $\rho_p = p/3$ . In the Bolgiano-Obhukhov regime, on the other hand, we expect the following dimensional values:  $\zeta_p = 3p/5$ ,  $\chi_p = p/5$ , and  $\rho_p = p/3$ .

In Fig. 3 we plot the behavior of the measured central exponents and their errors as a function of  $z$  for  $\zeta_3(z)$  and  $\chi_3(z)$ . The behavior is qualitatively and quantitatively consistent with the expected scaling exponents: we observe a smooth transition from a Bolgiano-Obhukhov dominated regime (near to the wall, small  $z$ ) to a Kolmogorov regime (approaching the center of the cell).

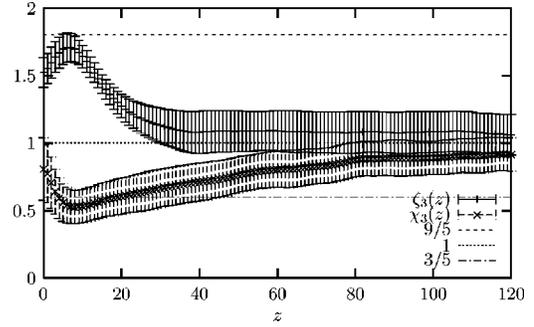


FIG. 3. Behavior of the exponents  $\zeta_3(z), \chi_3(z)$  as a function of  $z$ .

In Fig. 3 we have not shown the behavior of  $\rho_3(z)$  as it is consistent (within error bars) with the constant value 1 and plotting it on the same figure would have made it unreadable.

Once again we like to underline the consistency of our findings. From the theoretical picture we expect to see the crossover to the Kolmogorov scaling when  $L_B(z) \geq H$ . This is indeed the case as  $L_B(z) \geq H$  for  $z \geq 40$  and indeed we see that  $\zeta_3(z) \sim 1$  in the same range of  $z$ .

Another interesting question concerns the “real” statistics of the Bolgiano-Obhukhov regime. Very recently, interest is growing along this line of research in order to understand the differences between the statistical properties of an active with respect to a passive scalar [5,6]. In this work we focused on the gross features, i.e., on the dimensional behavior. If we try to look at intermittency, by means of ESS we find a strong increase of intermittency for the velocity field approaching the isothermal walls.

Unfortunately we believe that with our simulation we are not in a position to make any definite statements about intermittency in the Bolgiano-Obhukhov dominated regime.

Indeed a study of intermittency in the Bolgiano-Obhukhov regime would involve positions nearby the isothermal walls (as only there the Bolgiano length scale is small enough to have an inertial range largely dominated by buoyancy effects). Recently it was found that intermittency increases in the velocity structure functions inside a viscous boundary layer [12]. In an attempt to reduce the viscous boundary layer thickness we decided to apply stress-free velocity boundary conditions to the isothermal walls. This choice helps only partially because of two reasons. First, the stress-free boundary conditions does not completely suppress the boundary layer (for the nature of the resulting boundary layer, see, for example, [13]). A boundary layer thickness can be defined, as in Ref. [13], through an extrapolation of  $\langle v_z(z)^2 \rangle$ . In our case, it turned out to be roughly 15 grid spacings. Second, it was recently realized that mechanical turbulence in a stress-free (i.e., free-slip) channel also presents an enhancement of intermittency near to the walls [13].

A procedure to disentangle buoyancy and planar effects is clearly needed to make any definite statement on intermittency in the Bolgiano-Obhukhov regime. Otherwise one cannot decide whether a change of intermittency is related to Bolgiano-Obhukhov dynamics or to the increase occurring nearby boundary layers.

In order to clarify this point it would be important to

perform a simulation of a Rayleigh-Bénard-like system with periodic boundary conditions in all directions (i.e., an homogeneous Rayleigh-Bénard cell). We suggest to extend to 3D the study made in two dimensions in Ref. [5].

In order to achieve this one could write the temperature field as the sum of a linear profile plus a fluctuating part,  $T(x, y, z) = T_{\text{lin}}(z) + T'(x, y, z)$ , with  $T_{\text{lin}} = \Delta T/2 \times (1 - 2z/H)$  and obtain

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \alpha g T \hat{z}, \quad (12)$$

$$\partial_t T' + (\mathbf{v} \cdot \nabla) T' = \chi \nabla^2 T' + \frac{\Delta T}{H} v_z. \quad (13)$$

By choosing the parameters in order to have a Bolgiano length as small as possible, one would benefit of a wide

range of scales where to study the buoyancy dominated flow. Further advantage of homogeneity would be the natural increase of statistics and also the applicability of tools such as SO(3) decomposition, to disentangle anisotropic terms [15,16]. A study of this kind is in progress.

Concluding, we have performed a number of basic tests in order to validate the scenario of Bolgiano-Kolmogorov scaling in a convective cell, within the limitations but also the advantage of nonhomogeneity along  $z$  of our cell. We were able to confirm the transition between the two expected scenarios.

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